# Machine-Learning Error Models for Approximate Solutions to Parameterized Systems of Nonlinear Equations

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# Outline

- Introduction
- Parameterized Systems of Nonlinear Equations
- Machine-Learning Error Models
- Numerical Experiments
- Summary

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- Introduction
  - Motivation
  - Solution Approximations
  - Uncertainty Quantification
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# Motivation

- Many-query problems can impose a formidable computational burden
- **Solution approximations** can exchange fidelity for speed

# Solution Approximations

- **Inexact solutions:** When solving nonlinear equations, prematurely terminate iterations
- **Lower-fidelity models:** Neglect physical phenomena, coarsen the mesh, or use lower-order finite differences or elements
- **Reduced-order models:** Approximate solution with a linear combination of $m_{\mathbf{u}} \ll N_{\mathbf{u}}$ basis functions

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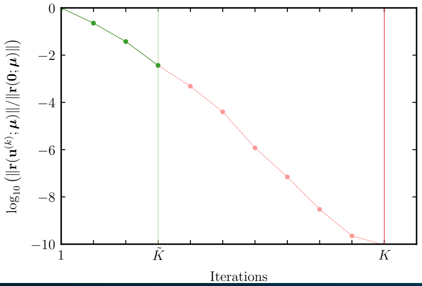

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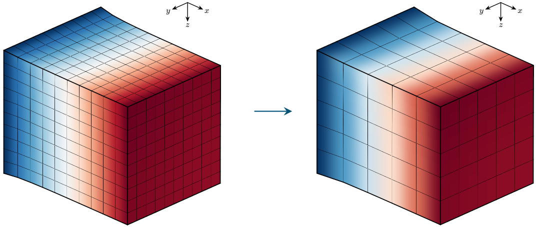

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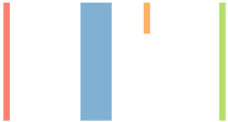
$$\tilde{\mathbf{u}}(\boldsymbol{\mu}) = \boldsymbol{\Phi}_{\mathbf{u}} \hat{\mathbf{u}}(\boldsymbol{\mu}) + \bar{\mathbf{u}}$$



# Uncertainty Quantification

- Solution approximations require less time than high-fidelity models but introduce an error (i.e., epistemic uncertainty)
- Ultimate task should account for **all sources of uncertainty**
- We quantify the uncertainty by
  1) **engineering features** informative of the error
     - cheaply computable
     - generated by approximate model
  2) applying **machine learning regression** techniques to construct a mapping from these features to a distribution of the error
- This work matures our previously developed capabilities:
  - Hand-selecting one feature and applying Gaussian process regression
    M. Drohmann and K. Carlberg (2015)
  - Modeling dynamical systems error using machine learning methods
    S. Trehan et al. (2017)

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  - Approaches for Error Quantification
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# Parameterized Systems of Nonlinear Equations

- Parameterized systems of nonlinear equations

$$\mathbf{r}(\mathbf{u}(\boldsymbol{\mu});\boldsymbol{\mu}) = \mathbf{0}$$

  - $\mathbf{r} : \mathbb{R}^{N_\mathbf{u}} \times \mathbb{R}^{N_{\boldsymbol{\mu}}} \rightarrow \mathbb{R}^{N_\mathbf{u}}$ residual, nonlinear in at least $\mathbf{u}(\boldsymbol{\mu})$
  - $\mathbf{u} : \mathbb{R}^{N_{\boldsymbol{\mu}}} \rightarrow \mathbb{R}^{N_\mathbf{u}}$ state (solution vector)
  - $\boldsymbol{\mu} \in \mathcal{D}$ parameters in parameter domain $\mathcal{D} \subseteq \mathbb{R}^{N_{\boldsymbol{\mu}}}$

- Scalar-valued quantity of interest

$$s(\boldsymbol{\mu}) := g(\mathbf{u}(\boldsymbol{\mu}))$$

  - $s : \mathbb{R}^{N_{\boldsymbol{\mu}}} \rightarrow \mathbb{R}$ quantity of interest
  - $g : \mathbb{R}^{N_\mathbf{u}} \rightarrow \mathbb{R}$ quantity of interest functional

# Approximate Solutions

- Computing the exact solution $\mathbf{u}(\boldsymbol{\mu})$ can be
  - prohibitively expensive (large $N_{\mathbf{u}}$)
  - unnecessary (inexact solutions suffice for optimization convergence)
- Such cases require an approximate solution $\tilde{\mathbf{u}} : \mathbb{R}^{N_{\boldsymbol{\mu}}} \rightarrow \mathbb{R}^{N_{\mathbf{u}}}$
- Approximate solution leads to approximated quantity of interest

$$\tilde{s}(\boldsymbol{\mu}) := g(\tilde{\mathbf{u}}(\boldsymbol{\mu})),$$

where $\tilde{s} : \mathbb{R}^{N_{\boldsymbol{\mu}}} \rightarrow \mathbb{R}$

# Approximate Solutions (continued)

We consider 3 approaches for computing approximate solutions:

1) Inexact solutions
2) Lower-fidelity models
3) Model reduction

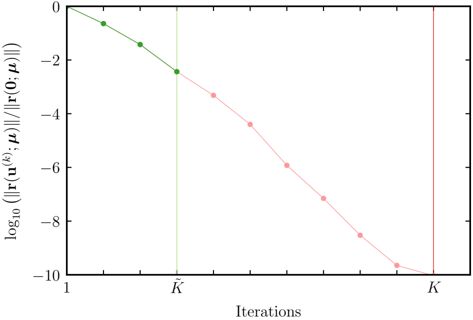
# Inexact Solutions

- Iterative solution to nonlinear equations: sequence of approximations

$$\mathbf{u}^{(k)}, \quad k = 0, \dots, K$$

- Approximate solution $\mathbf{u}^{(\tilde{K})}$ can be obtained after iteration $\tilde{K}$

$$\tilde{\mathbf{u}}(\boldsymbol{\mu}) = \mathbf{u}^{(\tilde{K})}$$

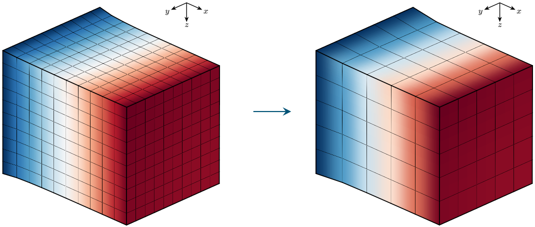


# Lower-Fidelity Models

## Fidelity reduction approaches

- Neglect physical phenomena
- Reduce spatial fidelity
  - Use lower-order finite differences or elements
  - Coarsen the mesh and prolongate (interpolate) the solution:

$$\tilde{\mathbf{u}} = \mathbf{p}(\mathbf{u}_{\mathrm{LF}}), \quad \mathbf{p} : \mathbb{R}^{N_{\mathbf{u}_{\mathrm{LF}}}} \rightarrow \mathbb{R}^{N_{\mathbf{u}}}$$



# Model Reduction

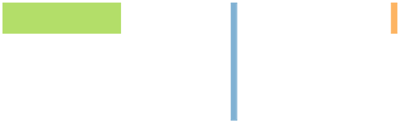
Model reduction restricts approximate solution $\tilde{\mathbf{u}}$ to $m_{\mathbf{u}}$-dimensional affine trial subspace $\text{Ran}(\boldsymbol{\Phi}_{\mathbf{u}}) + \bar{\mathbf{u}} \subseteq \mathbb{R}^{N_{\mathbf{u}}}$ with $m_{\mathbf{u}} \ll N_{\mathbf{u}}$:

$$\tilde{\mathbf{u}}(\boldsymbol{\mu}) = \boldsymbol{\Phi}_{\mathbf{u}} \hat{\mathbf{u}}(\boldsymbol{\mu}) + \bar{\mathbf{u}}$$

- $\boldsymbol{\Phi}_{\mathbf{u}} \in \mathbb{R}_{\star}^{N_{\mathbf{u}} \times m_{\mathbf{u}}}$ trial basis
- $\hat{\mathbf{u}} : \mathbb{R}^{N_{\boldsymbol{\mu}}} \to \mathbb{R}^{m_{\mathbf{u}}}$ generalized coordinates of approximate solution
- $\bar{\mathbf{u}} \in \mathbb{R}^{N_{\mathbf{u}}}$ prescribed reference state

Second step projects residual onto an $m_{\mathbf{u}}$-dimensional test subspace $\text{Ran}(\boldsymbol{\Psi}_{\mathbf{u}}) \subseteq \mathbb{R}^{N_{\mathbf{u}}}$:

$$\boldsymbol{\Psi}_{\mathbf{u}}^T \mathbf{r}(\boldsymbol{\Phi}_{\mathbf{u}} \hat{\mathbf{u}}(\boldsymbol{\mu}) + \bar{\mathbf{u}}; \boldsymbol{\mu}) = \mathbf{0}$$



- $\boldsymbol{\Psi}_{\mathbf{u}} \in \mathbb{R}_{\star}^{N_{\mathbf{u}} \times m_{\mathbf{u}}}$ test basis

# Approaches for Error Quantification

- Essential to quantify error incurred by approximate solution
- Existing approaches include
  - Data-fit mapping between parameters and the error
  - Reduced-Order Model Error Surrogates (ROMES) method

    M. Drohmann and K. Carlberg, 2015
    - Quantity-of-interest error approximated using dual-weighted residuals
    - Normed state-space error approx. using residual norm and error bounds
- We focus on quantifying two errors:
  1) Error in quantity of interest: $\delta_s(\boldsymbol{\mu}) := s(\boldsymbol{\mu}) - \tilde{s}(\boldsymbol{\mu})$
  2) Normed state-space error: $\delta_{\mathbf{u}}(\boldsymbol{\mu}) := \|\mathbf{e}(\boldsymbol{\mu})\|_2$, where $\mathbf{e}(\boldsymbol{\mu}) := \mathbf{u}(\boldsymbol{\mu}) - \tilde{\mathbf{u}}(\boldsymbol{\mu})$

# Error in Quantity of Interest: Dual-Weighted Residual

Approximate residual about approximate solution $\tilde{\mathbf{u}}$:

$$\mathbf{r}(\mathbf{u}(\boldsymbol{\mu});\boldsymbol{\mu}) = \mathbf{0} = \underbrace{\mathbf{r}(\tilde{\mathbf{u}}(\boldsymbol{\mu});\boldsymbol{\mu})}_{\mathbf{r}(\boldsymbol{\mu})} + \underbrace{\frac{\partial \mathbf{r}}{\partial \mathbf{v}}(\tilde{\mathbf{u}}(\boldsymbol{\mu});\boldsymbol{\mu})}_{\mathbf{J}(\boldsymbol{\mu})}\underbrace{(\mathbf{u}(\boldsymbol{\mu}) - \tilde{\mathbf{u}}(\boldsymbol{\mu}))}_{\mathbf{e}(\boldsymbol{\mu})} + \mathcal{O}(\|\mathbf{e}(\boldsymbol{\mu})\|^2)$$

Rearrange to approximate state-space error: $\mathbf{e}(\boldsymbol{\mu}) = -\mathbf{J}(\boldsymbol{\mu})^{-1}\mathbf{r}(\boldsymbol{\mu}) + \mathcal{O}(\|\mathbf{e}(\boldsymbol{\mu})\|^2)$ (1)

Approximate quantity of interest about $\tilde{\mathbf{u}}$: $s(\boldsymbol{\mu}) = \tilde{s}(\boldsymbol{\mu}) + \dfrac{\partial g}{\partial \mathbf{v}}(\tilde{\mathbf{u}}(\boldsymbol{\mu}))\mathbf{e}(\boldsymbol{\mu}) + \mathcal{O}(\|\mathbf{e}(\boldsymbol{\mu})\|^2)$

Combine with state-space error approximation (1):

$$\delta_s(\boldsymbol{\mu}) = \underbrace{-\frac{\partial g}{\partial \mathbf{v}}(\tilde{\mathbf{u}}(\boldsymbol{\mu}))\mathbf{J}(\boldsymbol{\mu})^{-1}}_{\mathbf{y}(\boldsymbol{\mu})^T\text{: dual or adjoint}}\mathbf{r}(\boldsymbol{\mu}) + \mathcal{O}(\|\mathbf{e}(\boldsymbol{\mu})\|^2)$$

Dual-weighted residual $d$ is weighted sum of residual elements:

$$d(\boldsymbol{\mu}) := \mathbf{y}(\boldsymbol{\mu})^T\mathbf{r}(\boldsymbol{\mu}) = \sum_{i=1}^{N_\mathbf{u}} y_i(\boldsymbol{\mu})r_i(\boldsymbol{\mu})$$

# Drawbacks to using the Dual-Weighted Residual

- **Computational Cost:** requires solving $N_{\mathbf{u}}$ linear equations
- **Implementation:** requires Jacobian – not always available
- **Uncertainty Quantification:** low-bias error estimate not assured

Nonetheless, structure provides insight into quantity-of-interest error

# Normed State-Space Error

- Residual-based bounds commonly used *a posteriori* to quantify $\delta_{\mathbf{u}}(\boldsymbol{\mu})$

  A. Buffa et al., 2012; M. A. Grepl and A. T. Patera, 2005; G. Rozza et al., 2008

- Assuming Lipschitz continuity for the residual $\mathbf{r}(\cdot;\boldsymbol{\mu})$, then

$$\frac{\|\mathbf{r}(\boldsymbol{\mu})\|}{\beta(\boldsymbol{\mu})} \leq \delta_{\mathbf{u}}(\boldsymbol{\mu}) \leq \frac{\|\mathbf{r}(\boldsymbol{\mu})\|}{\alpha(\boldsymbol{\mu})},$$

  where $\alpha$ and $\beta$ are Lipschitz constants

- Drawbacks to using error bounds
  - **Sharpness:** upper/lower bounds can overpredict/underpredict actual error by several orders of magnitude
  - **Implementation:** difficult to compute true Lipschitz constants
  - **Uncertainty Quantification:** do not produce statistical distribution over $\delta_{\mathbf{u}}(\boldsymbol{\mu})$ – cannot quantify epistemic uncertainty

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# Overview

- We aim to construct statistical models of
  - quantity-of-interest error $\delta_s$
  - normed state-space error $\delta_{\mathbf{u}}$
- We apply high-dimensional regression methods from machine learning
- We use a larger number of inexpensive error indicators, resulting in less costly, more accurate error models

# Error Model

- Assume there exist $N_{\mathbf{x}}$ *error indicators* or *features* $\mathbf{x}(\boldsymbol{\mu}) \in \mathbb{R}^{N_{\mathbf{x}}}$
  - available from solution approximation
  - cheaply computable
  - informative of the error $\delta(\boldsymbol{\mu}) \in \mathbb{R}$
- We model the nondeterministic mapping $\mathbf{x}(\boldsymbol{\mu}) \mapsto \delta(\boldsymbol{\mu})$

$$\delta(\boldsymbol{\mu}) = f(\mathbf{x}(\boldsymbol{\mu})) + \epsilon(\mathbf{x}(\boldsymbol{\mu}))$$

- $f$: deterministic regression function
- $\epsilon$: stochastic noise
  - Mean-zero random variable
  - Accounts for irreducible error due to omitted explanatory variables
  - Epistemic – additional features can enable zero noise

# Regression Model

- Regression function defines conditional expectation of error given the features:

$$\mathrm{E}[\delta(\boldsymbol{\mu}) \,|\, \mathbf{x}(\boldsymbol{\mu})] = f(\mathbf{x}(\boldsymbol{\mu}))$$

- We construct models of
  - deterministic regression function $\hat{f}(\approx f)$
  - stochastic noise $\hat{\epsilon}(\approx \epsilon)$,

  which yield a statistical model for the approximate-solution error

$$\underbrace{\hat{\delta}(\boldsymbol{\mu})}_{\text{stochastic}} = \underbrace{\hat{f}(\mathbf{x}(\boldsymbol{\mu}))}_{\text{deterministic}} + \underbrace{\hat{\epsilon}(\mathbf{x}(\boldsymbol{\mu}))}_{\text{stochastic}}$$

# Regression Model Objectives

- **Low Cost:** Should employ cheaply computable features $\mathbf{x}$
- **Low Noise Variance:** Should exhibit low noise variance, reduce epistemic uncertainty introduced by approximate solution
- **Generalize:** Empirical distributions of $\hat{\delta}$ and $\delta$ should be close on test set **not** used to train model – should not overfit on training data

# Regression Model Construction Steps

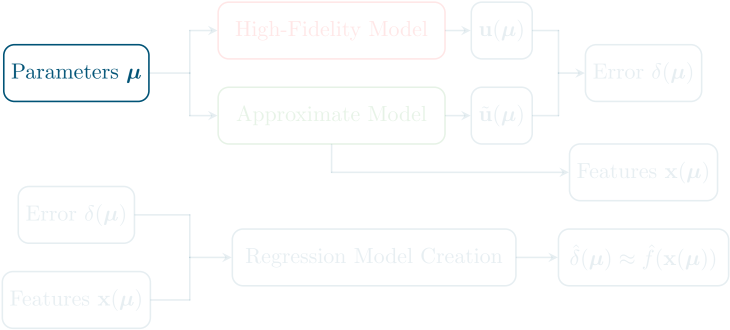
1) **Feature engineering**
   - Cheaply computable features $\mathbf{x}$ from approximate model
   - Informative of the error – construct low-noise-variance model
   - Low dimensional (small $N_{\mathbf{x}}$) such that less training data are needed

2) **Regression-function approximation**
   - Construct $\hat{f}$ using regression methods from machine learning
   - Approximate mapping from features $\mathbf{x}$ to error $\delta$ using a training set

3) **Noise approximation**
   - Mean-zero, constant-variance Gaussian random variable: $\hat{\epsilon} \sim \mathcal{N}(0, \hat{\sigma}^2)$
   - $\hat{\sigma}^2$ is sample variance of regression-model noise on a test set (mean squared error on test set)

# Summary

**Training**




**Application**

# Summary

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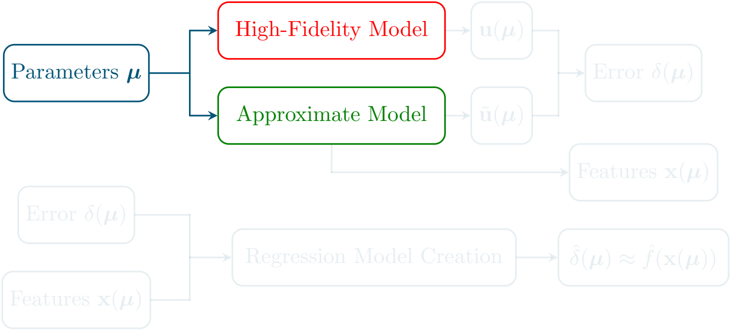
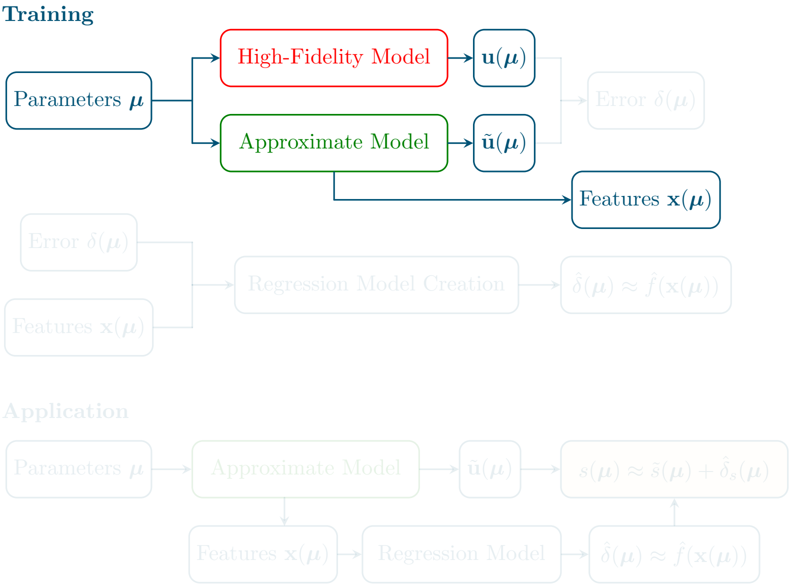



## Application

# Summary


Training
Parameters $\boldsymbol{\mu}$
High-Fidelity Model
$\mathbf{u}(\boldsymbol{\mu})$
Approximate Model
$\tilde{\mathbf{u}}(\boldsymbol{\mu})$
Error $\delta(\boldsymbol{\mu})$
Features $\mathbf{x}(\boldsymbol{\mu})$
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Regression Model Creation
$\hat{\delta}(\boldsymbol{\mu}) \approx \hat{f}(\mathbf{x}(\boldsymbol{\mu}))$
Application
Parameters $\boldsymbol{\mu}$
Approximate Model
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$s(\boldsymbol{\mu}) \approx \tilde{s}(\boldsymbol{\mu}) + \hat{\delta}_s(\boldsymbol{\mu})$
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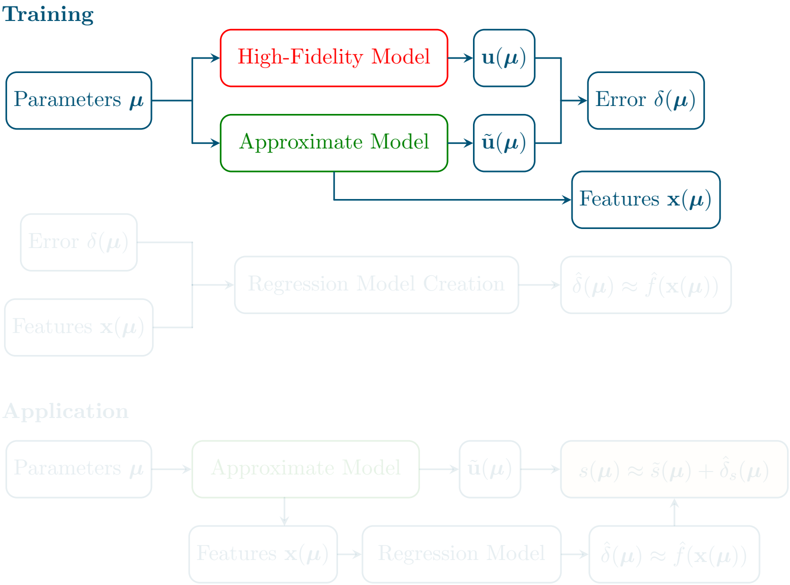
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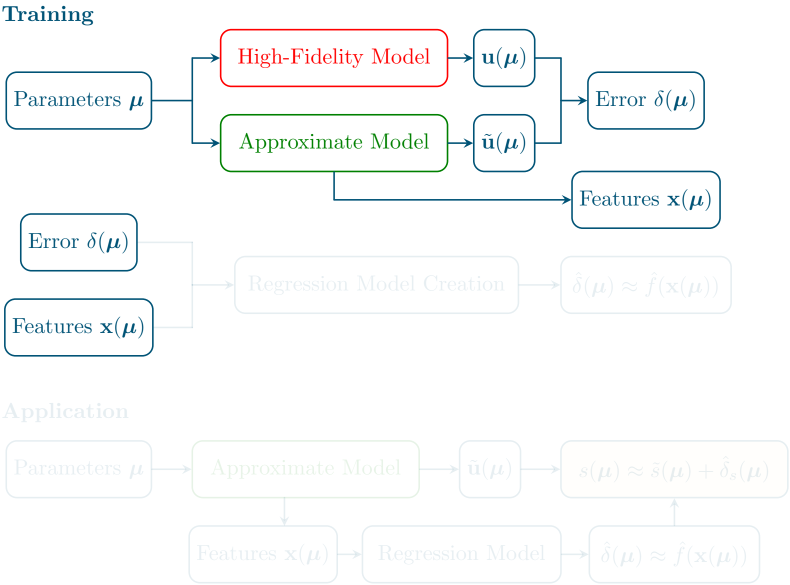
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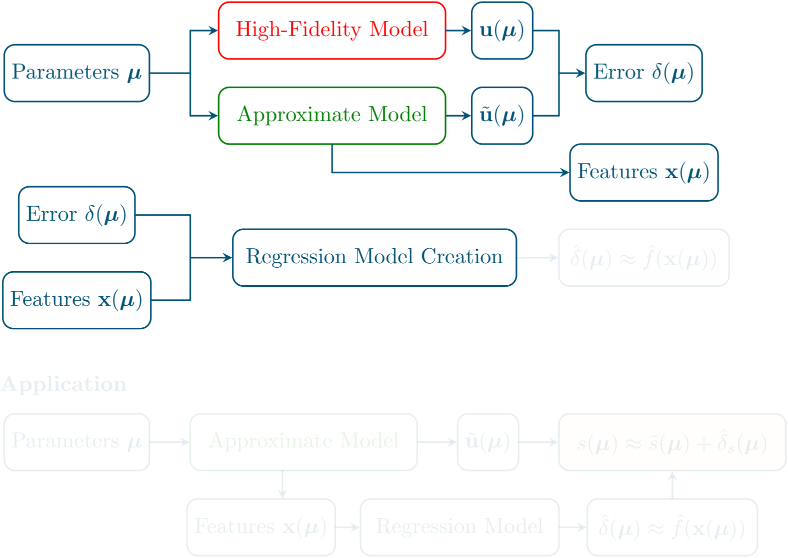

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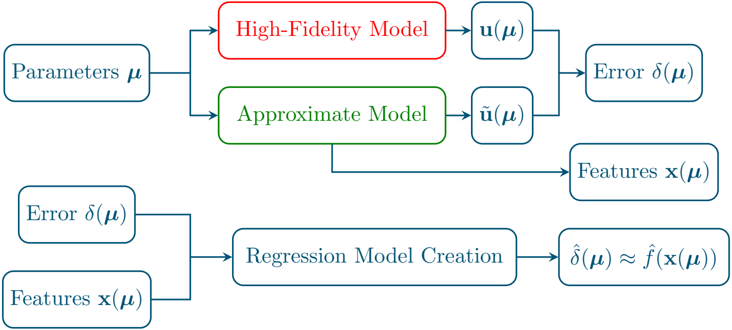
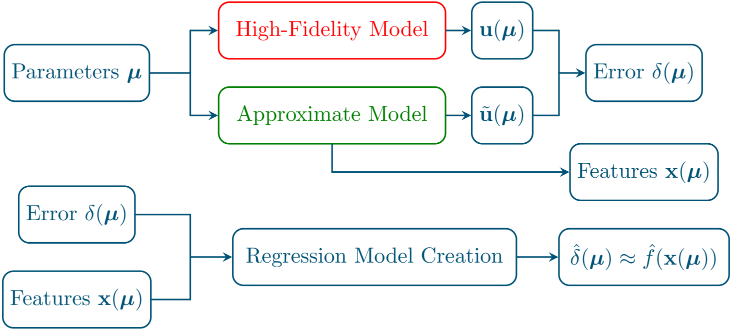



## Application

# Summary

### Training

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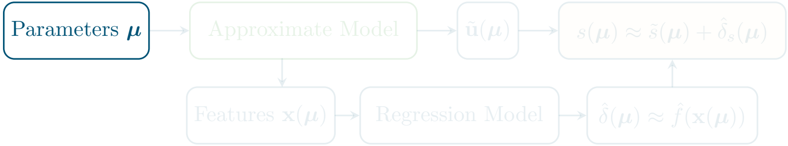

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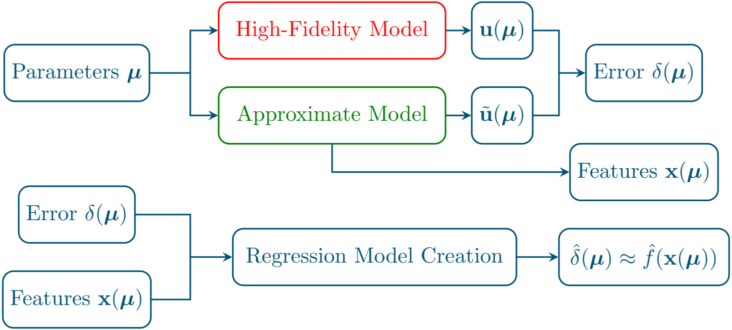


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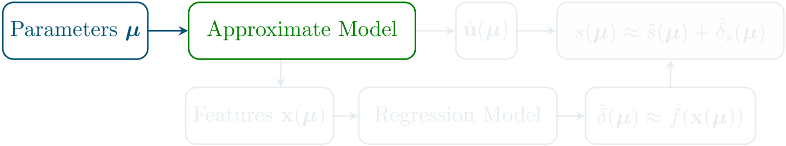

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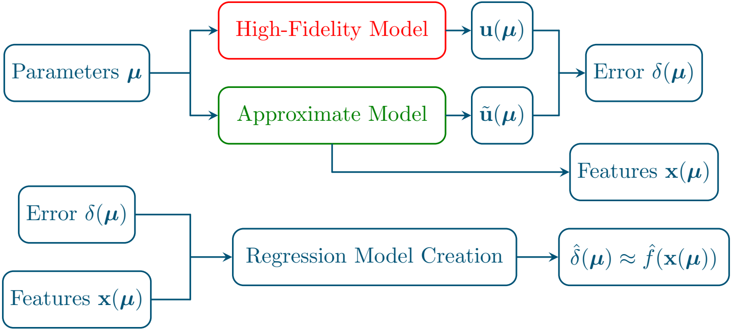


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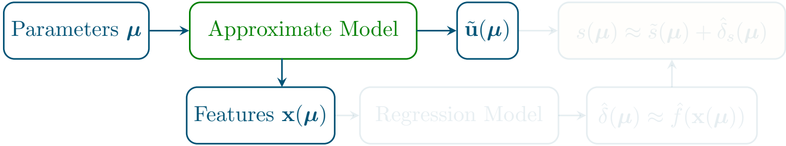

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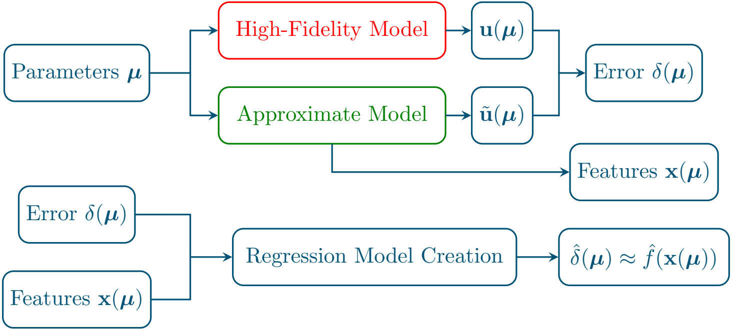


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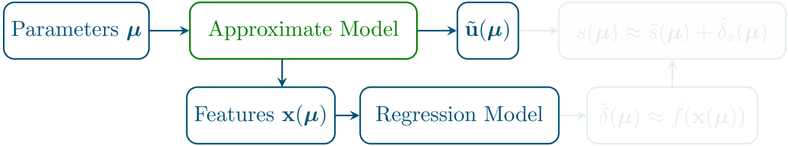

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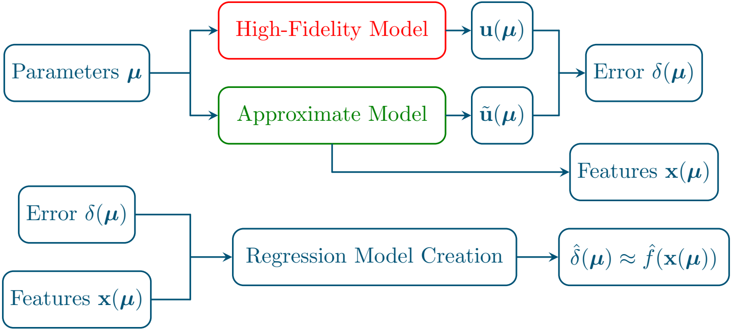


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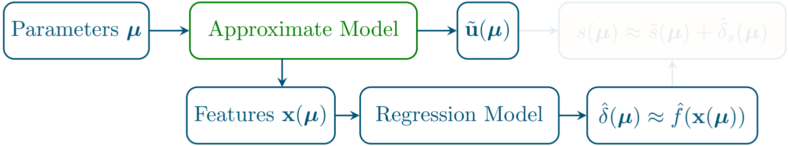

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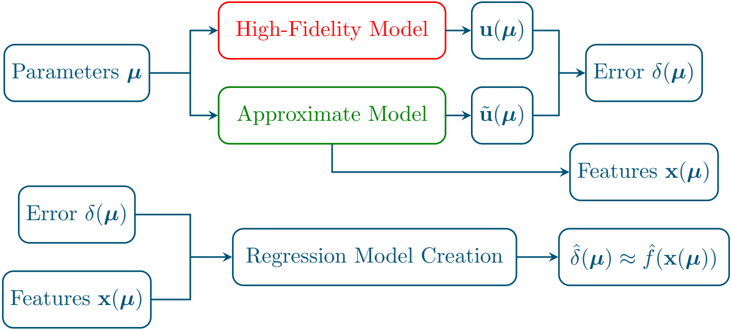


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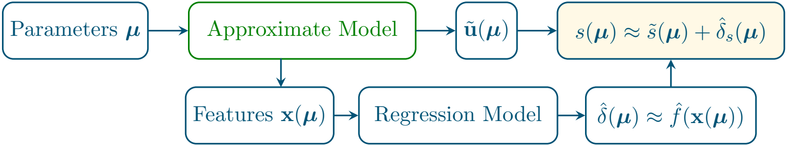


# Feature Engineering: Parameters

$$\mathbf{x}(\boldsymbol{\mu}) = \boldsymbol{\mu}$$

- The mapping $\boldsymbol{\mu} \mapsto \delta(\boldsymbol{\mu})$ is deterministic, but often complex
  - Can be oscillatory for ROMs since $\delta(\boldsymbol{\mu}) \approx 0$ when $\boldsymbol{\mu} \in \mathcal{D}_{\text{Train}}^{\text{ROM}}$
- Could yield zero noise variance if
  - Large amount of training data
  - High-capacity regression model
- Typically low-quality features
- Inspired by 'multifidelity correction' methods for optimization

  Alexandrov et al., 2001; Gano et al., 2005; Eldred et al., 2004

# Feature Engineering: Dual-Weighted Residual

$$\mathbf{x}(\boldsymbol{\mu}) = d(\boldsymbol{\mu}) := \mathbf{y}(\boldsymbol{\mu})^T \mathbf{r}(\boldsymbol{\mu})$$

- First-order approximation of QoI error $\delta_s(\boldsymbol{\mu})$
- Small number ($N_{\mathbf{x}} = 1$) of high-quality features
- High computational cost and significant implementation effort
- ROMES method uses approximation for dual-weighted residual

  M. Drohmann and K. Carlberg, 2015

# Feature Engineering: Parameters and Residual (Approximations)

$$\mathbf{x}(\boldsymbol{\mu}) = [\boldsymbol{\mu}; \mathbf{r}(\boldsymbol{\mu})]$$

- DWR is weighted sum of residual vector elements $d(\boldsymbol{\mu}) := \mathbf{y}(\boldsymbol{\mu})^T \mathbf{r}(\boldsymbol{\mu})$
- Avoids implementation and costs associated with dual vector $\mathbf{y}(\boldsymbol{\mu})$
- Large number ($N_{\mathbf{x}} = N_{\boldsymbol{\mu}} + N_{\mathbf{u}}$) of low-quality features
- Approaches to reduce number of features and improve quality
  - Use $m_{\mathbf{r}} \ll N_{\mathbf{u}}$ principal component coefficients: $\hat{\mathbf{r}}(\boldsymbol{\mu})$
  - Sample $n_{\mathbf{r}} \ll N_{\mathbf{u}}$ elements of residual: $\mathbf{P}\mathbf{r}(\boldsymbol{\mu})$, where $\mathbf{P} \in \{0,1\}^{n_{\mathbf{r}} \times N_{\mathbf{u}}}$
  - Use $m_{\mathbf{r}} \ll N_{\mathbf{u}}$ gappy principal component coefficients: $\hat{\mathbf{r}}_{\mathrm{g}}(\boldsymbol{\mu})$

# Feature Engineering: Residual Norm with/without Parameters

$$\mathbf{x}(\boldsymbol{\mu}) = \|\mathbf{r}(\boldsymbol{\mu})\|_2 \qquad \text{or} \qquad \mathbf{x}(\boldsymbol{\mu}) = [\boldsymbol{\mu};\ \|\mathbf{r}(\boldsymbol{\mu})\|_2]$$

- DWR can be bounded using the Cauchy–Schwarz inequality:

$$|d(\boldsymbol{\mu})| \leq \|\mathbf{y}(\boldsymbol{\mu})\|_2 \|\mathbf{r}(\boldsymbol{\mu})\|_2$$

- Normed state-space error $\delta_{\mathbf{u}}(\boldsymbol{\mu})$ can be bounded:

  M. Drohmann and K. Carlberg, 2015

$$\frac{\|\mathbf{r}(\boldsymbol{\mu})\|}{\beta(\boldsymbol{\mu})} \leq \delta_{\mathbf{u}}(\boldsymbol{\mu}) \leq \frac{\|\mathbf{r}(\boldsymbol{\mu})\|}{\alpha(\boldsymbol{\mu})}$$

- $\boldsymbol{\mu}$ can be omitted ($\mathbf{x}(\boldsymbol{\mu}) = \|\mathbf{r}(\boldsymbol{\mu})\|_2$) if
  - $\boldsymbol{\mu}$ is not indicative of error
  - $N_{\boldsymbol{\mu}}$ is too large relative to training data
- Requires computing entire residual vector $\mathbf{r}(\boldsymbol{\mu})$
- Small number of potentially low-quality features

# Regression-Function Approximation

We consider several different regression models

- Ordinary least squares (OLS)
  - Linear (OLS: Linear)
  - Quadratic expansion of features (OLS: Quadratic)
- Support vector regression (SVR)
  - Linear kernel (SVR: Linear)
  - Gaussian (radial basis function) kernel (SVR: RBF)
- Random forest (RF)
- $k$-nearest neighbors ($k$-NN)
- Artificial neural network (ANN)

# Training and Test Data

## Training Data

- Set of parameter training instances $\mathcal{D}_{\text{train}} \subset \mathcal{D}$
- Train regression models from high-fidelity and approx. solutions
  - Cross-validated to tune regression-model hyper-parameters
- Used to compute principal components of residuals

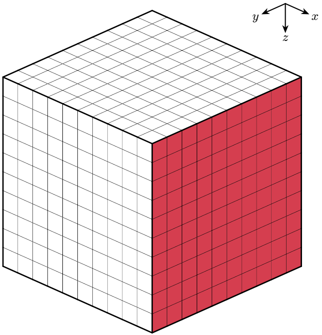
## Test Data

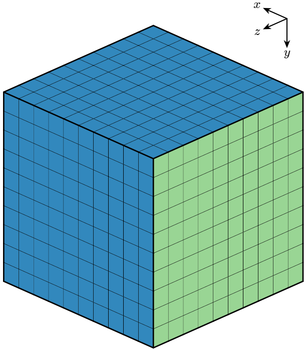
- Set of parameter test instances $\mathcal{D}_{\text{test}} \subset \mathcal{D}$ **not** used for training ($\mathcal{D}_{\text{train}} \cap \mathcal{D}_{\text{test}} = \emptyset$)
- Used to assess regression models and quantify stochastic noise

# Outline

- Introduction
- Parameterized Systems of Nonlinear Equations
- Machine-Learning Error Models
- Numerical Experiments
  - Cube: Reduced-Order Modeling
  - PCAP: Reduced-Order Modeling
  - Burgers' Equation: Inexact Solutions and Coarse Solution Prolongation
- Summary

# Cube: Reduced-Order Modeling

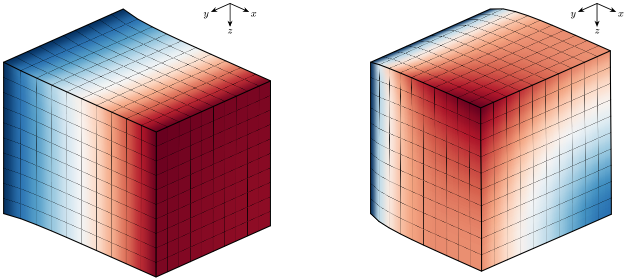


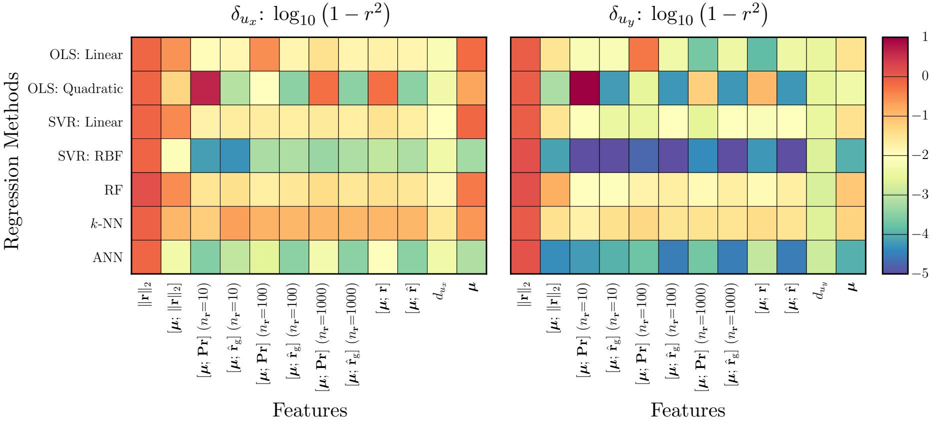

- Applied traction (Neumann boundary condition)
- Planar constraint (Dirichlet boundary condition)
- Complete constraint (Dirichlet boundary condition)
- Node of interest

# Cube: Overview

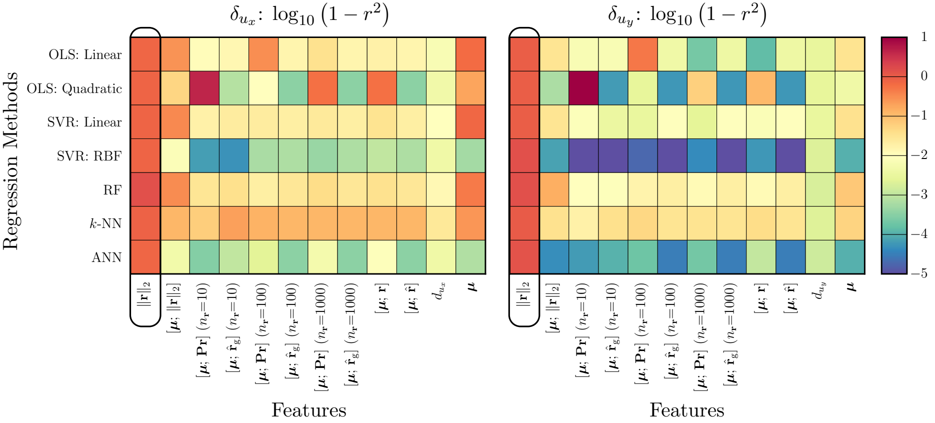
- $N_{\mathbf{u}} = 3410$ – deliberately small to compute $d(\boldsymbol{\mu})$ and use $\mathbf{r}(\boldsymbol{\mu})$
- $N_{\boldsymbol{\mu}} = 3$ parameters: $\boldsymbol{\mu} = [E;\, \nu;\, t]$
  - $E \in [75,\, 125]$ GPa, $\nu \in [0.20,\, 0.35]$, $t \in [40,\, 60]$ GPa
- 8 HF runs $\rightarrow$ up to $m_{\mathbf{u}} = 8$ ROM basis vectors (2 used – 99.49%)



# Cube: Variance Unexplained for QoI Error Prediction

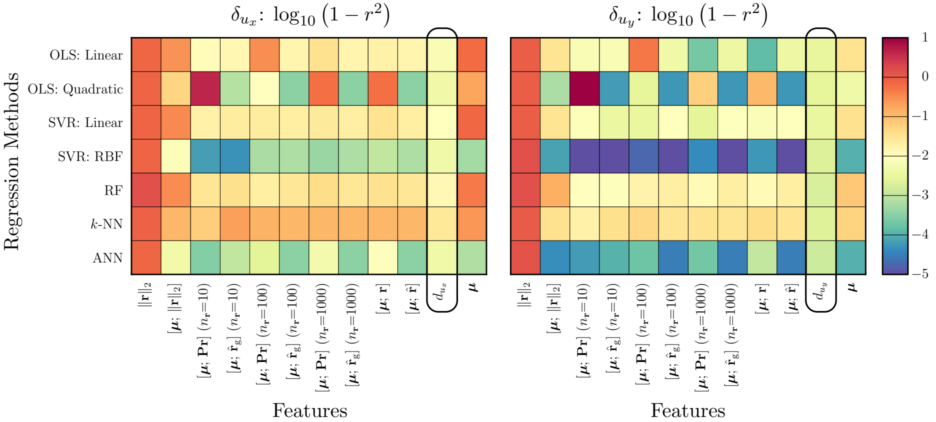


# Cube: Variance Unexplained for QoI Error Prediction



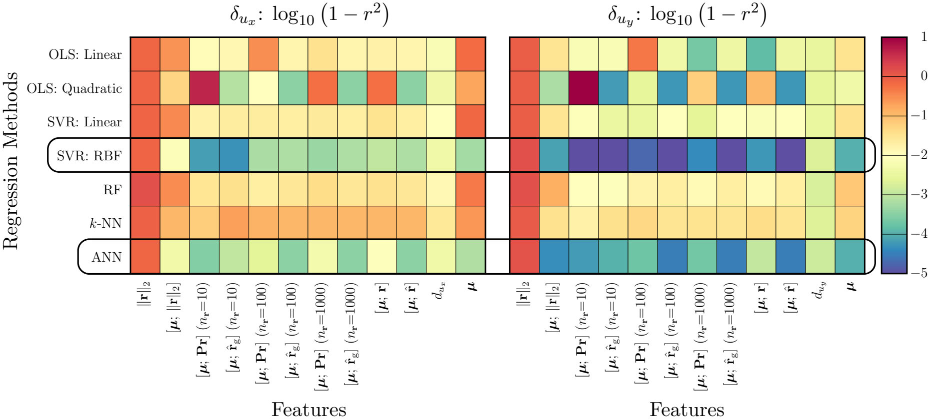

- $\|\mathbf{r}\|_2$ yields highest variance unexplained

# Cube: Variance Unexplained for QoI Error Prediction



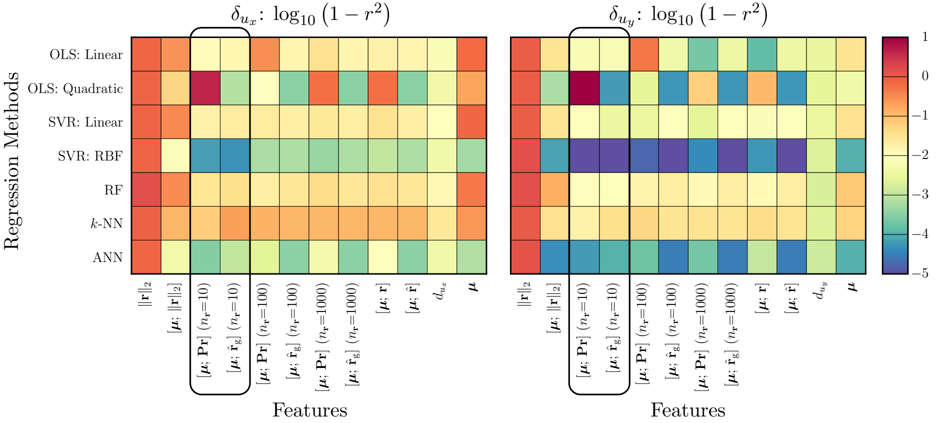

- $\|\mathbf{r}\|_2$ yields highest variance unexplained
- $d_{u_x}$ and $d_{u_y}$ yield moderate variance unexplained, but are costly

# Cube: Variance Unexplained for QoI Error Prediction



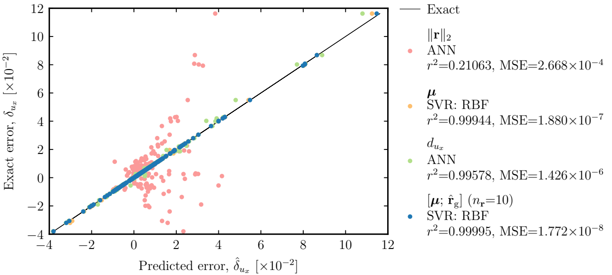

- $\|\mathbf{r}\|_2$ yields highest variance unexplained
- $d_{u_x}$ and $d_{u_y}$ yield moderate variance unexplained, but are costly
- SVR: RBF and ANN yield lowest variance unexplained

# Cube: Variance Unexplained for QoI Error Prediction

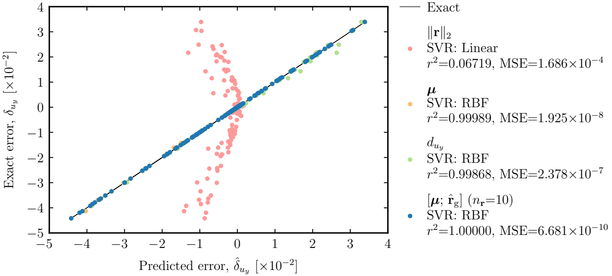



- $\|\mathbf{r}\|_2$ yields highest variance unexplained
- $d_{u_x}$ and $d_{u_y}$ yield moderate variance unexplained, but are costly
- SVR: RBF and ANN yield lowest variance unexplained
- $[\boldsymbol{\mu}; \hat{\mathbf{r}}_\mathrm{g}]$ and $[\boldsymbol{\mu}; \mathbf{Pr}]$ yield low variance unexplained with only 10 samples (compared to $N_\mathbf{u} = 3410$)

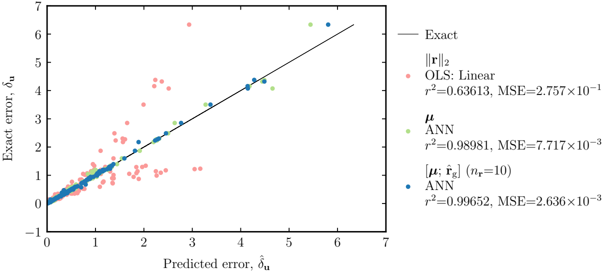
# Cube: QoI Error Predictions




- Our method beats previous state-of-the-art methods with $r^2 > 0.9999$ in both cases

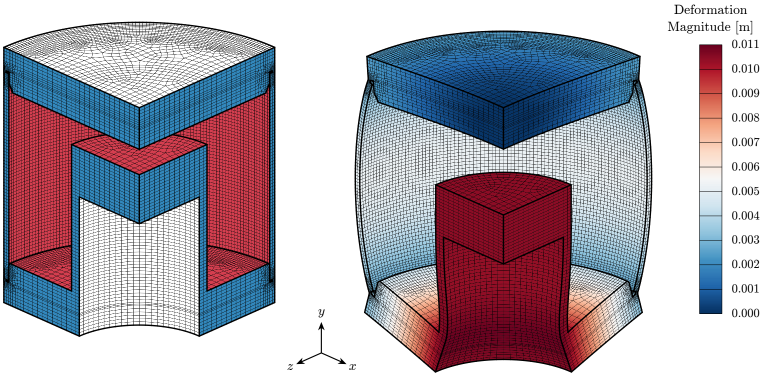


# Cube: Normed State-Space Error Predictions




- Our method beats previous state-of-the-art methods with $r^2 > 0.996$

# Predictive Capability Assessment Project: Reduced-Order Modeling

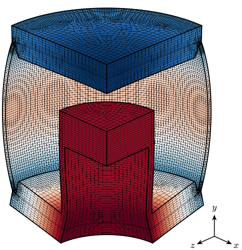



- Applied pressure (Neumann boundary condition)
- Planar constraint (Dirichlet boundary condition)
- Complete constraint (Dirichlet boundary condition)
- Nodes of interest

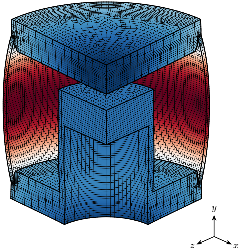
# PCAP: Overview

- $N_{\mathbf{u}} = 274,954$ for quarter of domain
- $N_{\boldsymbol{\mu}} = 3$ parameters: $\boldsymbol{\mu} = [E;\, \nu;\, p]$
  - $E \in [50,\ 100]$ GPa, $\nu \in [0.20,\ 0.35]$, $p \in [6,\ 10]$ GPa
- 8 HF runs $\rightarrow$ up to $m_{\mathbf{u}} = 8$ ROM basis vectors (5 used – 99.90%)
- 30 parameter training instances for regression model

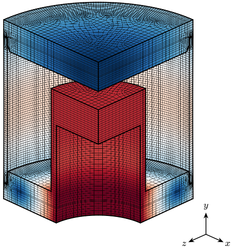
# PCAP: Basis Vectors



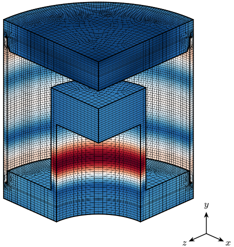

1: 85.03%



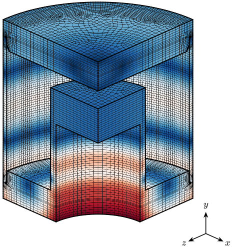

2: 95.69%




3: 99.35%

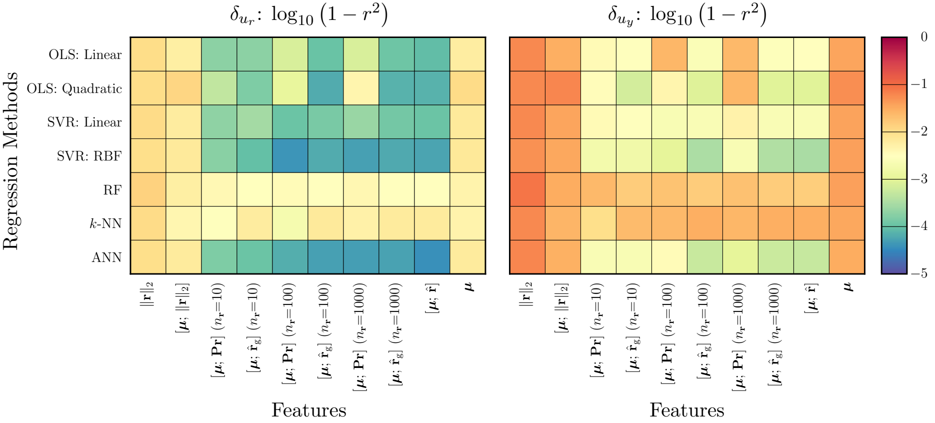



4: 99.77%

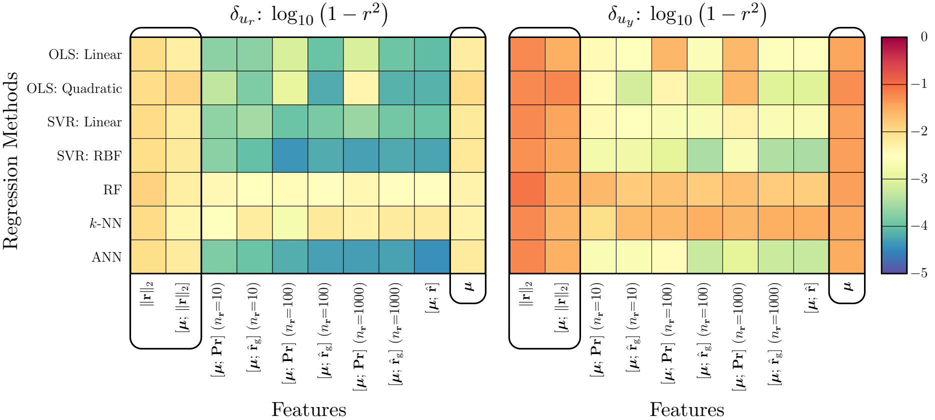



5: 99.90%

# PCAP: Variance Unexplained for QoI Error Prediction

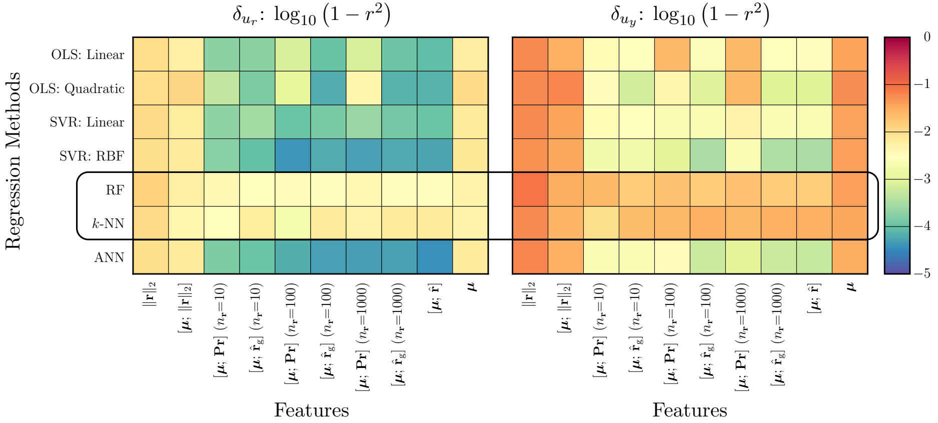


# PCAP: Variance Unexplained for QoI Error Prediction



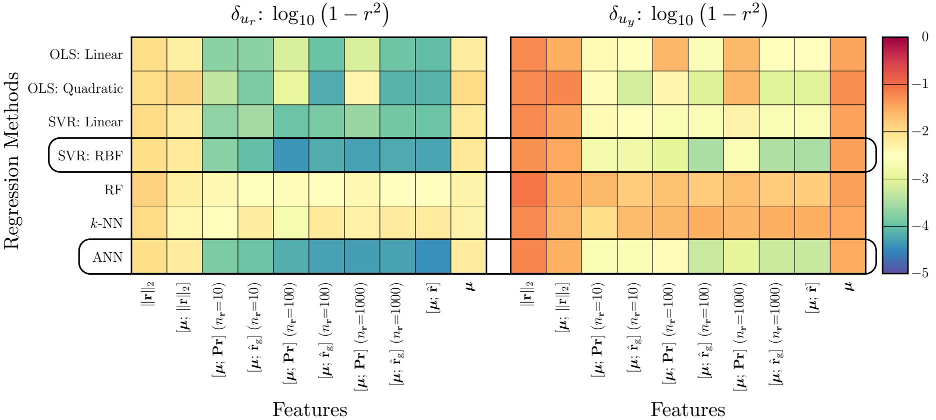

- $\|\mathbf{r}\|_2$, $[\boldsymbol{\mu}; \|\mathbf{r}\|_2]$, and $\boldsymbol{\mu}$ yield highest variance unexplained

# PCAP: Variance Unexplained for QoI Error Prediction



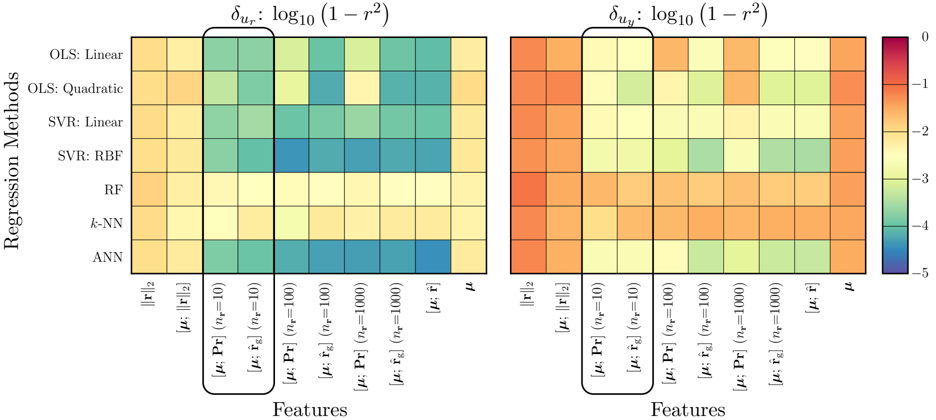

- $\|\mathbf{r}\|_2$, $[\boldsymbol{\mu}; \|\mathbf{r}\|_2]$, and $\boldsymbol{\mu}$ yield highest variance unexplained
- RF and $k$-NN yield highest variance unexplained

# PCAP: Variance Unexplained for QoI Error Prediction




- $\|\mathbf{r}\|_2$, $[\boldsymbol{\mu}; \|\mathbf{r}\|_2]$, and $\boldsymbol{\mu}$ yield highest variance unexplained
- RF and $k$-NN yield highest variance unexplained
- SVR: RBF and ANN yield lowest variance unexplained

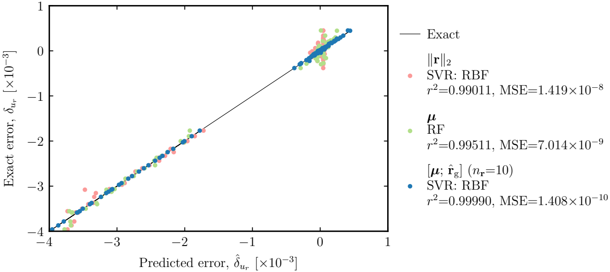
# PCAP: Variance Unexplained for QoI Error Prediction

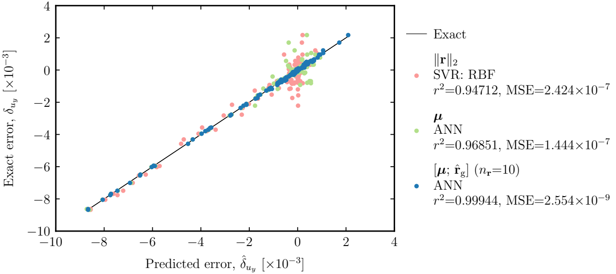



- $\|\mathbf{r}\|_2$, $[\boldsymbol{\mu}; \|\mathbf{r}\|_2]$, and $\boldsymbol{\mu}$ yield highest variance unexplained
- RF and $k$-NN yield highest variance unexplained
- SVR: RBF and ANN yield lowest variance unexplained
- $[\boldsymbol{\mu}; \hat{\mathbf{r}}_\text{g}]$ and $[\boldsymbol{\mu}; \mathbf{Pr}]$ yield low variance unexplained with only 10 samples (compared to $N_\mathbf{u} = 274,954$)

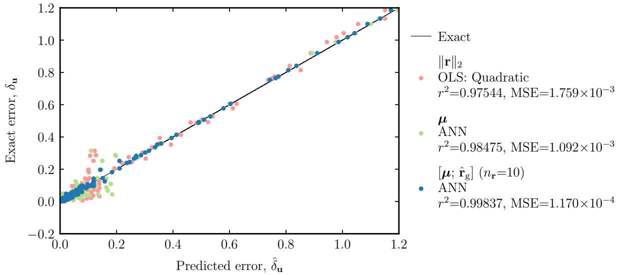
# PCAP: QoI Error Predictions

- Our method beats previous state-of-the-art methods with $r^2 > 0.9994$ in both cases



# PCAP: Normed State-Space Error Predictions

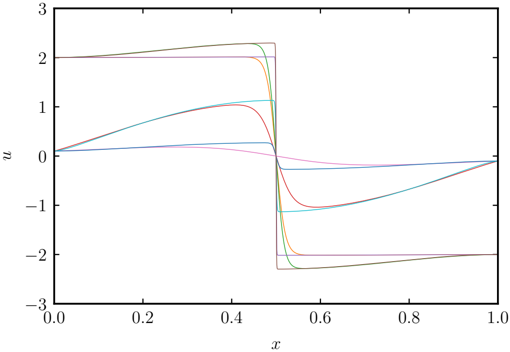



- Our method beats previous state-of-the-art methods with $r^2 > 0.998$

# Burgers' Equation: Inexact Solutions and Coarse Solution Prolongation

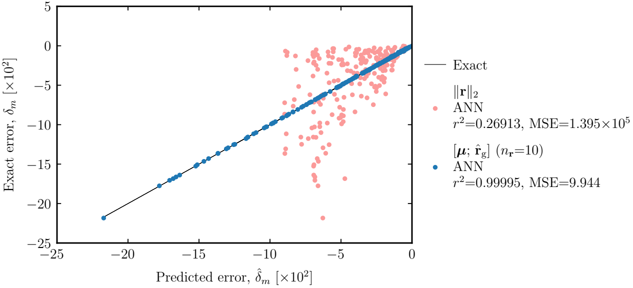
$$uu_x - \frac{1}{R}u_{xx} = \alpha \sin 2\pi x$$

$$u(0) = u_a, \quad u(1) = -u_a$$



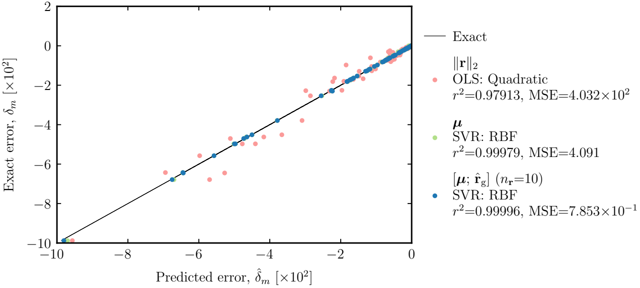

- $N_{\mathbf{u}} = 1999$
- $N_{\boldsymbol{\mu}} = 3$ parameters: $\boldsymbol{\mu} = [\alpha;\, u_a;\, R]$
  - $\alpha \in [0.1,\, 2.0]$, $\quad u_a \in [0.1,\, 2.0]$, $\quad R \in [50,\, 1000]$
- Quantity of interest $s$ is the slope $m$ at $x = \frac{1}{2}$
- $\tilde{K} = 1$ and $\tilde{K} = 2$ or $N_{\mathbf{u}_{\mathrm{LF}}} = 499$ and $N_{\mathbf{u}_{\mathrm{LF}}} = 999$

# Burgers' Equation, Inexact Solutions: QoI Error Predictions




- Our method beats previous state-of-the-art method with $r^2 > 0.9999$

# Burgers' Equation, Coarse Mesh Prolongation: QoI Error Predictions




- Our method beats previous state-of-the-art methods with $r^2 > 0.9999$

# Outline

- Introduction
- Parameterized Systems of Nonlinear Equations
- Machine-Learning Error Models
- Numerical Experiments
- Summary
  - Feature Choices
  - Feature Reduction

# Feature Choices

- Norm of the residual, $\|\mathbf{r}\|_2$
  - Low-quality single feature
  - Expensive to compute and performs poorly
- Dual-weighted residual, $d$
  - High-quality single feature
  - Performs well for small amounts of training data
  - Very expensive to compute
- Parameters $\boldsymbol{\mu}$
  - Only perform well with SVR: RBF or ANN
  - Do not perform well with OLS: Linear
- Parameters and gappy principal components of residual, $[\boldsymbol{\mu}; \hat{\mathbf{r}}_\text{g}]$
  - Perform the best with $r^2 > 0.996$ for each experiment
  - Only require about 13 features

# Feature Reduction

- Gappy PCA more effective than directly sampling the residual
- Little benefit to using $n_{\mathbf{r}} \geq 100$ samples; more samples are more expensive and do not perform much better
- Often, only $n_{\mathbf{r}} = 10$ samples are necessary to get accurate prediction

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# Questions?

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