# Manufactured Solutions for an Electromagnetic Slot Model


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# Outline

- Introduction
- Governing Equations
- Code-Verification Approaches
- Numerical Examples
- Summary

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- Introduction
  - Electromagnetic Integral Equations
  - Verification and Validation
  - Error Sources
  - This Work
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- Code-Verification Approaches
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# Electromagnetic Integral Equations

- Are commonly used to model electromagnetic scattering and radiation
- Relate surface current to incident electric and/or magnetic field
- Discretize surface of electromagnetic scatterer with elements
- Evaluate 4D reaction integrals over 2D test and source elements
- Contain singular integrands when test and source elements are near

# Electromagnetic Aperture and Slot Models

- EM penetration occurs through openings of otherwise closed surfaces
- Penetration may occur intentionally or unintentionally
- Slot connects exterior surface of scatterer to interior surface of cavity
- Model slot as wires carrying magnetic current located at apertures
  - Exterior surface interacts with exterior wire
  - Interior surface interacts with interior wire
  - Exterior and interior wires interact with each other
  - Exterior and interior surfaces do not interact directly

# Verification and Validation

Credibility of computational physics codes requires verification and validation

- **Validation** assesses how well models represent physical phenomena
  - Compare computational results with experimental results
  - Assess suitability of models, model error, and bounds of validity

- **Verification** assesses accuracy of numerical solutions against expectations
  - *Solution verification* estimates numerical error for particular solution
  - *Code verification* verifies correctness of numerical-method implementation

# Code Verification

- Code verification most rigorously assesses rate at which error decreases
- Error requires exact solution – usually unavailable
- Manufactured solutions are popular alternative
  - Manufacture an arbitrary solution
  - Insert manufactured solution into governing equations to get residual term
  - Add residual term to equations to coerce solution to manufactured solution
- For integral equations, few instances of code verification exist
- Analytical differentiation is straightforward – analytical integration is not
- Numerical integration is necessary, generally incurs an approximation error
- Therefore, manufactured source term may have its own numerical error

# Error Sources in the Electromagnetic Integral Equations

3 sources of numerical error:

- **Domain discretization**: Representation of curved surfaces with planar elements
  - Second-order error for curved surfaces, no error for planar surfaces
  - Error reduced with curved elements
- **Solution discretization**: Representation of solution or operators
  - Common in solution to differential, integral, and integro-differential equations
  - Finite number of basis functions to approximate solution
  - Finite samples queried to approximate underlying equation operators
- **Numerical integration**: Quadrature
  - Analytical integration is not always possible
  - For well-behaved integrands,
    - Expect integration error at least same order as solution-discretization error
    - Less rigorously, error should decrease with more quadrature points
  - For (nearly) singular integrands, monotonic convergence is not assured

# This Work

**Isolate solution-discretization error**

- Manufacture solution
- Eliminate numerical-integration error by manufacturing Green's function
- Mitigate contamination from discontinuity due to wire–surface interaction

**Isolate numerical-integration error**

- Manufacture solution
- Cancel solution-discretization error using basis functions

**Avoid domain-discretization error**

- Consider only planar surfaces
- Previously provided approaches to account for domain-discretization error

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  - The Electric-Field Integral Equation
  - The Slot Equation
  - Discretization
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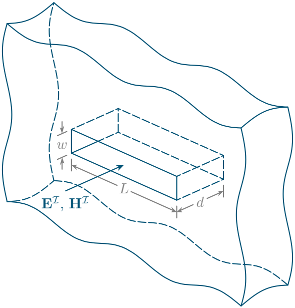
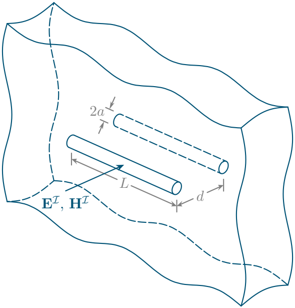
# Thick Slot Model Overview




- Electromagnetic scatterer encloses a cavity
- Exterior is connected to interior by rectangularly prismatic slot with $L \gg w, d$ (left)
- Slot is replaced with two thin wires at apertures that carry magnetic current (right)
- Exterior and interior surfaces interact with wires, not each other
- Wires interact with each other – magnetic current is equal and opposite
- EFIE solved on each surface, slot equation solved for wires

# The Electric-Field Integral Equation

In time-harmonic form, $\mathbf{E}^{\mathcal{S}}$ computed from $\mathbf{J}$ and $\mathbf{M}$

Scattered electric field $$\mathbf{E}^{\mathcal{S}}(\mathbf{x}) = -\left(j\omega\mathbf{A}(\mathbf{x}) + \nabla\Phi(\mathbf{x}) + \frac{1}{\epsilon}\nabla\times\mathbf{F}(\mathbf{x})\right)$$

Magnetic vector potential $$\mathbf{A}(\mathbf{x}) = \mu\int_{S'}\mathbf{J}(\mathbf{x}')G(\mathbf{x},\mathbf{x}')dS'$$

Electric scalar potential $$\Phi(\mathbf{x}) = \frac{j}{\epsilon\omega}\int_{S'}\nabla'\cdot\mathbf{J}(\mathbf{x}')G(\mathbf{x},\mathbf{x}')dS'$$

Electric vector potential $$\mathbf{F}(\mathbf{x}) = \epsilon\int_{S'}\mathbf{M}(\mathbf{x}')G(\mathbf{x},\mathbf{x}')dS'$$

Green's function $$G(\mathbf{x},\mathbf{x}') = \frac{e^{-jkR}}{4\pi R}, \qquad R = |\mathbf{x}-\mathbf{x}'|$$

Singularity when $R\to 0$

$\mathbf{J}$ and $\mathbf{M}$ are electric and magnetic surface current densities
$S' = S$ is surface of scatterer
$\mu$ and $\epsilon$ are permeability and permittivity of surrounding medium
$k = \omega\sqrt{\mu\epsilon}$ is wavenumber

# The Electric-Field Integral Equation (continued)

Compute $\mathbf{J}$ and $\mathbf{M}$ from incident electric field $\mathbf{E}^{\mathcal{I}}$ $\quad (\mathbf{n} \times (\mathbf{E}^{\mathcal{S}} + \mathbf{E}^{\mathcal{I}}) = Z_s \mathbf{n} \times \mathbf{J})$

Discretize surface with triangles, approximate $\mathbf{J}$ with RWG basis functions:

$$\mathbf{J}_h(\mathbf{x}) = \sum_{j=1}^{n_b} J_j \boldsymbol{\Lambda}_j(\mathbf{x})$$

Project EFIE onto vector-valued RWG basis functions

Express $\mathbf{M}$ in terms of filament magnetic current $\mathbf{I}_m$

Discretize wire with bars, approximate $\mathbf{I}_m$ with 1D basis functions:

$$\mathbf{I}_h(s) = \sum_{j=1}^{n_b^m} I_j \boldsymbol{\Lambda}_j^m(s)$$

# The Slot Equation

The magnetic current along the slot is modeled using transmission line theory:

$$\mathbf{s} \cdot \left[ \mathbf{J} \times \mathbf{n} + \frac{1}{4}\left( Y_L \frac{d^2}{ds^2} - Y_C \right) \mathbf{I}_m \right] = 0$$

$\mathbf{I}_m(0) = \mathbf{I}_m(L) = \mathbf{0}$

$\mathbf{s}$ is the direction of the wire

Effective wire radius $a$ obtained from conformal mapping using $w$ and $d$

$Y_L$ and $Y_C$ are transmission line parameters (depend on $w$, $d$, and materials)

Project slot equation onto vector-valued 1D basis functions

# Discretized Problem

Find $\mathbf{J}_h \in \mathbb{V}_h$ and $\mathbf{I}_h \in \mathbb{V}_h^m$, such that

$$a_{\mathcal{E},\mathcal{E}}(\mathbf{J}_h, \mathbf{\Lambda}_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_h, \mathbf{\Lambda}_i) = b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}}, \mathbf{\Lambda}_i) \qquad \text{for } i = 1, \dots, n_b \quad \text{(EFIE)}$$

$$a_{\mathcal{M},\mathcal{E}}(\mathbf{J}_h, \mathbf{\Lambda}_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_h, \mathbf{\Lambda}_i^m) = 0 \qquad \text{for } i = 1, \dots, n_b^m \quad \text{(Slot)}$$

Evaluate EFIE on exterior and interior surfaces: $n_b^{\text{ext}} + n_b^{\text{int}}$ unknowns for $\mathbf{J}_h$

For thick slot, $\mathbf{I}_m^{\text{ext}} = -\mathbf{I}_m^{\text{int}}$: $n_b^m$ unknowns for $\mathbf{I}_h$

# Matrix–Vector Form

In matrix–vector form, solve for $\mathcal{J}^h$:

$$\mathbf{Z}\mathcal{J}^h = \mathbf{V}$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{A}^{\text{ext}} & \mathbf{0} & \mathbf{B}^{\text{ext}} \\ \mathbf{0} & \mathbf{A}^{\text{int}} & -\mathbf{B}^{\text{int}} \\ \mathbf{C}^{\text{ext}} & -\mathbf{C}^{\text{int}} & \mathbf{D} \end{bmatrix}, \qquad \mathcal{J}^h = \begin{Bmatrix} \mathbf{J}^{h\,\text{ext}} \\ \mathbf{J}^{h\,\text{int}} \\ \mathbf{I}^h \end{Bmatrix}, \qquad \mathbf{V} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}\,\text{ext}} \\ \mathbf{V}^{\mathcal{E}\,\text{int}} \\ \mathbf{0} \end{Bmatrix},$$

Impedance matrix — Current vector — Excitation vector

where

$$A_{i,j} = a_{\mathcal{E},\mathcal{E}}(\boldsymbol{\Lambda}_j, \boldsymbol{\Lambda}_i), \quad B_{i,j} = a_{\mathcal{E},\mathcal{M}}(\boldsymbol{\Lambda}_j^m, \boldsymbol{\Lambda}_i), \quad C_{i,j} = a_{\mathcal{M},\mathcal{E}}(\boldsymbol{\Lambda}_j, \boldsymbol{\Lambda}_i^m), \quad D_{i,j} = 2a_{\mathcal{M},\mathcal{M}}(\boldsymbol{\Lambda}_j^m, \boldsymbol{\Lambda}_i^m),$$

$$J_j^h = J_j, \qquad I_j^h = I_j, \qquad V_j^{\mathcal{E}} = b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}}, \boldsymbol{\Lambda}_i)$$

More compactly:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \qquad \mathcal{J}^h = \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix}, \qquad \mathbf{V} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}} \\ \mathbf{0} \end{Bmatrix}$$

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  - Numerical-Integration Error
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# Manufactured Solutions for the EFIE

Continuous: $r_{\mathcal{E}_i}(\mathbf{J}\ \ ,\mathbf{I}_m) = a_{\mathcal{E},\mathcal{E}}(\mathbf{J}\ \ ,\boldsymbol{\Lambda}_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_m,\boldsymbol{\Lambda}_i) - b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}},\boldsymbol{\Lambda}_i) = 0$

Discretized: $r_{\mathcal{E}_i}(\mathbf{J}_h,\mathbf{I}_h\ ) = a_{\mathcal{E},\mathcal{E}}(\mathbf{J}_h,\boldsymbol{\Lambda}_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_h\ ,\boldsymbol{\Lambda}_i) - b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}},\boldsymbol{\Lambda}_i) = 0$

Method of manufactured solutions modifies discretized equations:

$$\mathbf{r}_{\mathcal{E}}(\mathbf{J}_h,\mathbf{I}_h) = \mathbf{r}_{\mathcal{E}}(\mathbf{J}_{\text{MS}},\mathbf{I}_{\text{MS}})$$

$\mathbf{J}_{\text{MS}}$ and $\mathbf{I}_{\text{MS}}$ are manufactured solutions, $\mathbf{r}_{\mathcal{E}}(\mathbf{J}_{\text{MS}},\mathbf{I}_{\text{MS}})$ is computed exactly

New Discretized: $a_{\mathcal{E},\mathcal{E}}(\mathbf{J}_h,\boldsymbol{\Lambda}_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_h,\boldsymbol{\Lambda}_i) = \underbrace{a_{\mathcal{E},\mathcal{E}}(\mathbf{J}_{\text{MS}},\boldsymbol{\Lambda}_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_{\text{MS}},\boldsymbol{\Lambda}_i)}_{= b_{\mathcal{E}}\left(\mathbf{E}^{\mathcal{I}},\boldsymbol{\Lambda}_i\right)\text{: implement via } \mathbf{E}^{\mathcal{I}}}$

$$\mathbf{E}^{\mathcal{I}}(\mathbf{x}) = \frac{j}{\epsilon\omega}\int_{S'}\left[k^2\mathbf{J}_{\text{MS}}(\mathbf{x}')G(\mathbf{x},\mathbf{x}') + \nabla'\cdot\mathbf{J}_{\text{MS}}(\mathbf{x}')\nabla G(\mathbf{x},\mathbf{x}')\right]dS' + Z_s\mathbf{J}_{\text{MS}}(\mathbf{x})$$
$$- \frac{1}{4}(\mathbf{n}(\mathbf{x})\times\mathbf{I}_{\text{MS}}(\mathbf{x}))\delta_{\text{slot}}(\mathbf{x}) + \frac{1}{4\pi}\int_0^L \mathbf{I}_{\text{MS}}(s')\times\int_0^{2\pi}\nabla' G(\mathbf{x},\mathbf{x}')d\phi' ds'$$

MMS incorporated through $\mathbf{E}^{\mathcal{I}}$ – no additional source term required

# Manufactured Solutions for the Slot Equation

Continuous: $r_{\mathcal{M}_i}(\mathbf{J}\ \ , \mathbf{I}_m) = a_{\mathcal{M},\mathcal{E}}(\mathbf{J}\ \ , \boldsymbol{\Lambda}_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_m, \boldsymbol{\Lambda}_i^m) = 0$

Discretized: $r_{\mathcal{M}_i}(\mathbf{J}_h, \mathbf{I}_h\ ) = a_{\mathcal{M},\mathcal{E}}(\mathbf{J}_h, \boldsymbol{\Lambda}_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_h\ , \boldsymbol{\Lambda}_i^m) = 0$

Method of manufactured solutions modifies discretized equations:

$$\mathbf{r}_{\mathcal{M}}(\mathbf{J}_h, \mathbf{I}_h) = \mathbf{r}_{\mathcal{M}}(\mathbf{J}_{\text{MS}}, \mathbf{I}_{\text{MS}})$$

New Discretized:

$$a_{\mathcal{M},\mathcal{E}}(\mathbf{J}_h, \boldsymbol{\Lambda}_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_h, \boldsymbol{\Lambda}_i^m) = \underbrace{a_{\mathcal{M},\mathcal{E}}(\mathbf{J}_{\text{MS}}, \boldsymbol{\Lambda}_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_{\text{MS}}, \boldsymbol{\Lambda}_i^m)}_{=0:\ \text{no source term needed}}$$

Given $\mathbf{J}_{\text{MS}}$, solve for $\mathbf{I}_{\text{MS}}$ to avoid source term

# Solution-Discretization Error

- Error due to basis-function approximations of solutions:

$$\mathbf{J}_h(\mathbf{x}) = \sum_{j=1}^{n_b} J_j \boldsymbol{\Lambda}_j(\mathbf{x}), \qquad \mathbf{I}_h(s) = \sum_{j=1}^{n_b^m} I_j \boldsymbol{\Lambda}_j^m(s)$$

- Measured with discretization errors: $\mathbf{e_J} = \mathbf{J}^h - \mathbf{J}_n, \quad \mathbf{e_I} = \mathbf{I}^h - \mathbf{I}_s$

$$\|\mathbf{e_J}\| \leq C_\mathbf{J} h^{p_\mathbf{J}}, \qquad \|\mathbf{e_I}\| \leq C_\mathbf{I} h^{p_\mathbf{I}}$$

$J_{n_j}$: component of $\mathbf{J}_{\text{MS}}$ flowing from $T_j^+$ to $T_j^-$
$I_{s_j}$: component of $\mathbf{I}_{\text{MS}}$ flowing along $\mathbf{s}$ at $s_j$
$C$ : function of solution derivatives
$h$ : measure of mesh size
$p$ : order of accuracy

- Compute $p$ from $\|\mathbf{e}\|$ across multiple meshes (expect $p = 2$ for these bases)

- Avoid numerical-integration error if integrating exactly

# Solution-Discretization Error: Discontinuity

- $\delta_{\text{slot}}$ introduces discontinuity due to wire interaction with surface
- Discontinuity impacts $\mathbf{E}^{\mathcal{I}}$ for MMS
- Discontinuity will contaminate convergence studies: $\mathcal{O}(h^2) \rightarrow \mathcal{O}(h)$
- Discontinuity denoted by $\mathbf{B}_1$ in $\mathbf{Z} = \begin{bmatrix} \mathbf{A} & (\mathbf{B}_1 + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$
- Since $\mathbf{B}_1 = -\frac{1}{4}\mathbf{C}^T$, use $\mathbf{C}$ to cancel contribution from $\mathbf{B}_1$ and modify $\mathbf{E}^{\mathcal{I}}$:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{A} & (\cancel{\mathbf{B}}_1 + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix},$$

$$\mathbf{E}^{\mathcal{I}} = \frac{j}{\epsilon\omega}\int_{S'}[k^2\mathbf{J}_{\text{MS}}G + \nabla'\cdot\mathbf{J}_{\text{MS}}\nabla G]dS' - \cancel{\frac{1}{4}(\mathbf{n}\times\mathbf{I}_{\text{MS}})\delta_{\text{slot}}} + \frac{1}{4\pi}\int_0^L \mathbf{I}_{\text{MS}} \times \int_0^{2\pi}\nabla' G d\phi' ds' + Z_s\mathbf{J}_{\text{MS}}$$

- Correctness of $\mathbf{B}_1$ is assessed by successful removal using $\mathbf{C}$
- Correctness of $\mathbf{C}$ is assessed through the mesh-convergence study

# Numerical-Integration Error

- Error due to quadrature integral evaluation $(\cdot)^q$ on both sides of equations
- Measure numerical-integration error:

$$e_a = \boldsymbol{\mathcal{J}}^H(\mathbf{Z}^q - \mathbf{Z})\boldsymbol{\mathcal{J}}, \qquad e_b = \boldsymbol{\mathcal{J}}^H(\mathbf{V}^q - \mathbf{V}),$$

where $\boldsymbol{\mathcal{J}} = \begin{Bmatrix} \mathbf{J}_n \\ \mathbf{I}_s \end{Bmatrix}$

- Solution-discretization error is canceled
- $|e_a| \leq C_a h^{p_a}$ and $|e_b| \leq C_b h^{p_b}$

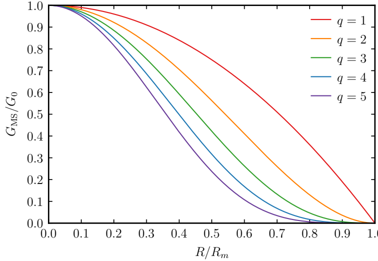
  $C$: function of integrand derivatives
  $p$: order of accuracy of quadrature rules
- With multiple meshes, compute $p$ from $|e|$

# Manufactured Green's Function

Integrals with $G$ cannot be computed analytically or, when $R \to 0$, accurately

Inaccurately computing integrals on either side contaminates convergence studies

Manufacture Green's function: $G_{\text{MS}}(R) = G_0\left(1 - \frac{R^2}{R_m^2}\right)^q$, $R_m = \max\limits_{\mathbf{x},\mathbf{x}'\in S} R$ and $q \in \mathbb{N}$




Reasoning:
1) Even powers of $R$ permit integrals to be computed analytically
2) $G_{\text{MS}}$ increases when $R$ decreases, as with actual $G$

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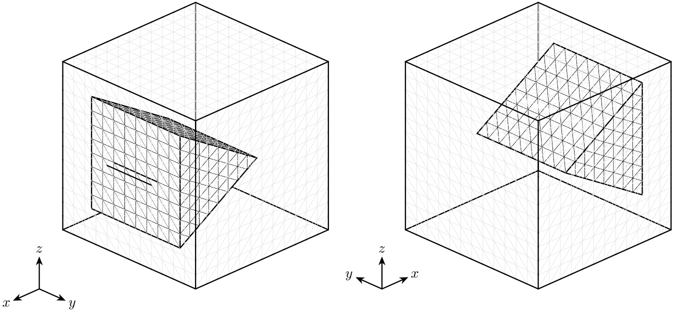
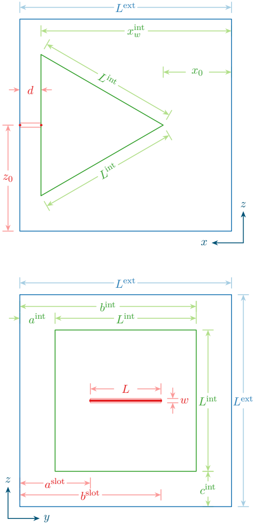
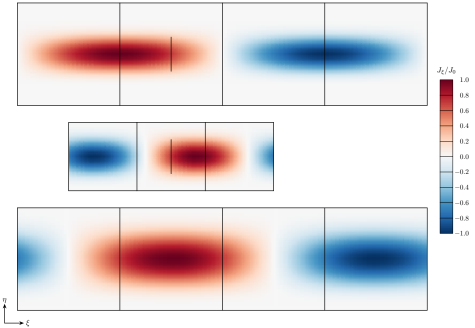
# Cubic Scatterer with a Triangularly Prismatic Cavity

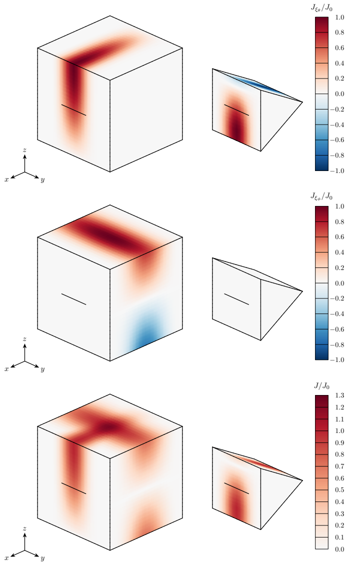



- $L^{\text{ext}} = 1$ m, $\quad L^{\text{int}} = 2L^{\text{ext}}/3, \quad L = L^{\text{ext}}/3, \quad w = L^{\text{ext}}/50$
- $a^{\text{int}} = L^{\text{ext}}/6, \quad c^{\text{int}} = L^{\text{ext}}/6, \quad z_0 = L^{\text{ext}}/2$
- $\mu = \mu_0, \quad \epsilon = \epsilon_0, \quad k = 2\pi$ m$^{-1}$, $\quad \sigma$ of aluminum
- 3 depths: $d_1 = L^{\text{ext}}/10, \quad d_2 = L^{\text{ext}}/100, \quad d_3 = L^{\text{ext}}/1000$
- 2 Green's functions: $G_1$, $G_2$

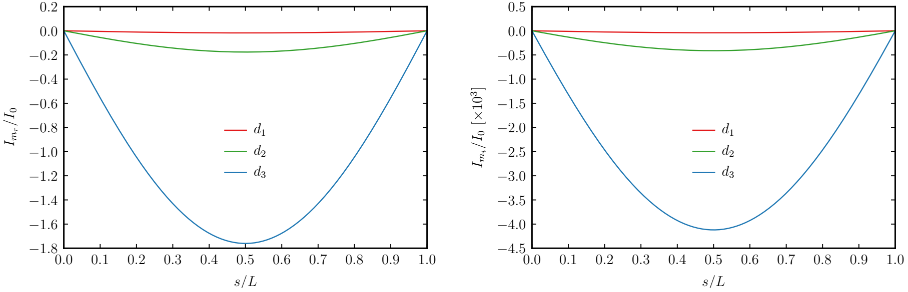
# Manufactured Solutions




- Manufacture solutions for 2D strips of class $C^2$
- Wrap strips around lateral surfaces of prisms
- Solutions are product of $\xi$ and $\eta$ dependencies
  - $\xi$ dependency: sinusoid with a single period
  - $\eta$ dependency: cubed sinusoid with a half period
- Current flows along $\xi$; at slot, $J_\xi^{\text{ext}} = J_\xi^{\text{int}}$

# Magnetic Current




- Instead of arbitrarily manufacturing $\mathbf{I}_{\text{MS}} = I_m\mathbf{s}$, solve for it given $\mathbf{J}_{\text{MS}}$:

$$-J_{\xi_\theta}(\boldsymbol{\xi}) + \frac{1}{4}\left(Y_L\frac{d^2}{ds^2} - Y_C\right)I_m(s) = 0$$

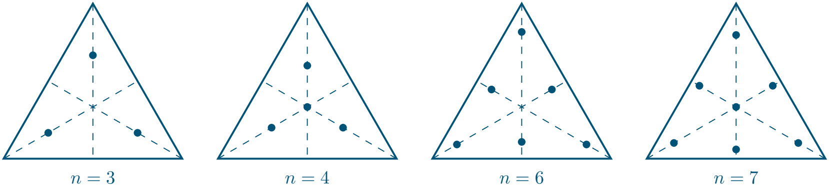
- Solution is

$$I_m(s) = C_1\cosh\left(\frac{s}{Z}\right) + C_2\sinh\left(\frac{s}{Z}\right) + C_3\sin\left(\frac{\pi(s+\Delta a)}{L^{\text{int}}}\right) + C_4\sin\left(\frac{3\pi(s+\Delta a)}{L^{\text{int}}}\right)$$

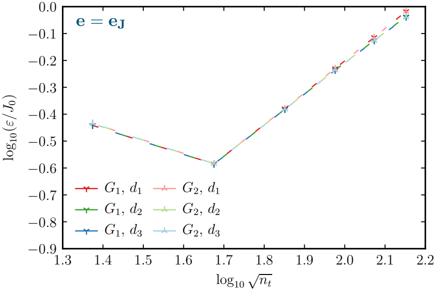
# Numerical Integration

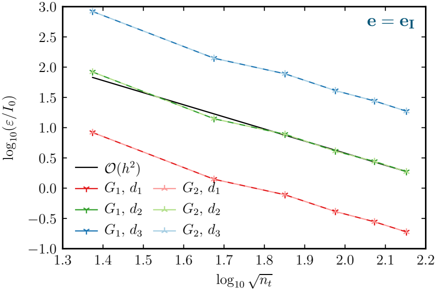
- Surface integrals evaluated using 2D triangle quadrature rules
- Wire integrals evaluated using 1D bar quadrature rules

| Maximum integrand degree | Number of 2D points | Number of 1D points | Convergence rate |
|---|---|---|---|
| 1 | 1 | 1 | $\mathcal{O}(h^2)$ |
| 2 | 3 | — | $\mathcal{O}(h^4)$ |
| 3 | 4 | 2 | $\mathcal{O}(h^4)$ |
| 4 | 6 | — | $\mathcal{O}(h^6)$ |
| 5 | 7 | 3 | $\mathcal{O}(h^6)$ |



# Solution-Discretization Error: $\varepsilon = \|\mathbf{e}\|_\infty$ $(\mathbf{e_J} \leftrightarrow \mathbf{e_I})$



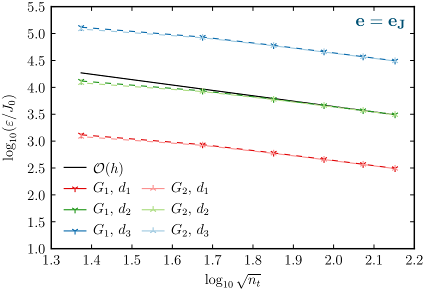



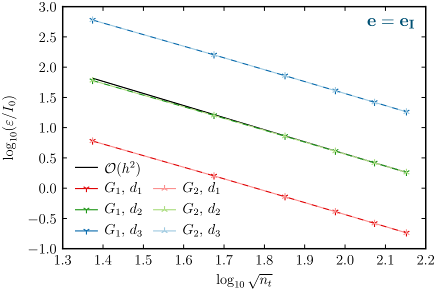

- Discontinuity present:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}} \\ \mathbf{0} \end{Bmatrix}$$

- $\mathbf{e_J}$ and $\mathbf{e_I}$ are interdependent $(\mathbf{e_J} \leftrightarrow \mathbf{e_I})$
- Convergence rate for $\|\mathbf{e_J}\|_\infty$ is negative – not exhibiting asymptotic convergence
- Convergence rate for $\|\mathbf{e_I}\|_\infty$ is close to $\mathcal{O}(h^2)$

# Solution-Discretization Error: $\varepsilon = \|\mathbf{e}\|_\infty$ ($\mathbf{e_J} \not\leftrightarrow \mathbf{e_I}$)



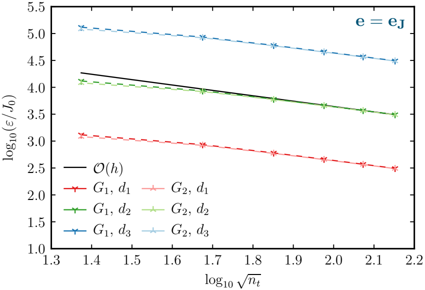



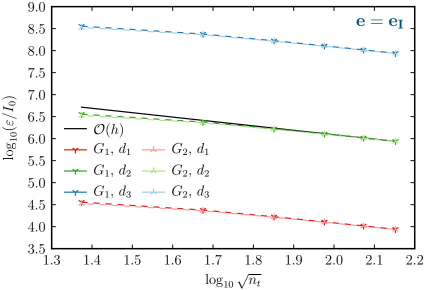

- Decouple discretization errors:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\mathcal{E} \\ \mathbf{0} \end{Bmatrix} \rightarrow \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\mathcal{E} - \mathbf{B}\mathbf{I}_s \\ -\mathbf{C}\mathbf{J}_n \end{Bmatrix}$$

- $\mathbf{e_J}$ and $\mathbf{e_I}$ are independent ($\mathbf{e_J} \not\leftrightarrow \mathbf{e_I}$)
- Convergence rates for $\|\mathbf{e_J}\|_\infty$ and $\|\mathbf{e_I}\|_\infty$ are $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$ as expected
- $\|\mathbf{e_J}\|_\infty$ is much larger than when $\mathbf{e_J} \leftrightarrow \mathbf{e_I}$

# Solution-Discretization Error: $\varepsilon = \|\mathbf{e}\|_\infty$ $(\mathbf{e_J} \rightarrow \mathbf{e_I})$



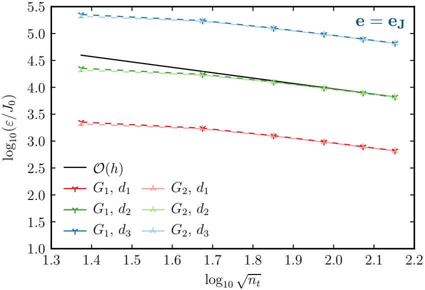



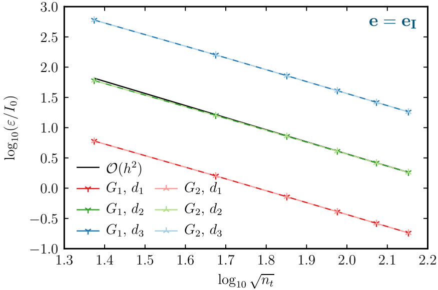

- Remove influence of $\mathbf{e_I}$ on $\mathbf{e_J}$, preserve influence of $\mathbf{e_J}$ on $\mathbf{e_I}$:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}} \\ \mathbf{0} \end{Bmatrix} \rightarrow \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}} - \mathbf{B}\mathbf{I}_s \\ \mathbf{0} \end{Bmatrix}$$

- $\mathbf{e_J}$ affects $\mathbf{e_I}$ $(\mathbf{e_J} \rightarrow \mathbf{e_I})$
- Convergence rates for $\|\mathbf{e_J}\|_\infty$ and $\|\mathbf{e_I}\|_\infty$ are $\mathcal{O}(h)$ as expected
- $\|\mathbf{e_J}\|_\infty$ and $\|\mathbf{e_I}\|_\infty$ are much larger than when $\mathbf{e_J} \leftrightarrow \mathbf{e_I}$

# Solution-Discretization Error: $\varepsilon = \|\mathbf{e}\|_\infty$ $(\mathbf{e_J} \leftarrow \mathbf{e_I})$



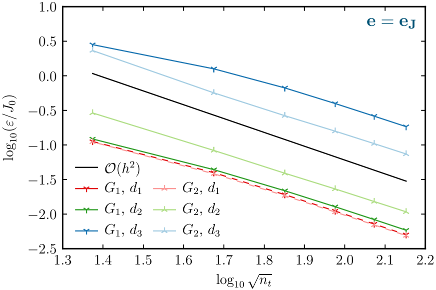



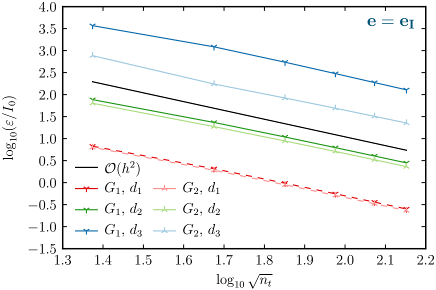

- Remove influence of $\mathbf{e_J}$ on $\mathbf{e_I}$, preserve influence of $\mathbf{e_I}$ on $\mathbf{e_J}$:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}} \\ \mathbf{0} \end{Bmatrix} \rightarrow \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}} \\ -\mathbf{C}\mathbf{J}_n \end{Bmatrix}$$

- $\mathbf{e_I}$ affects $\mathbf{e_J}$ $(\mathbf{e_J} \leftarrow \mathbf{e_I})$
- Convergence rates for $\|\mathbf{e_J}\|_\infty$ and $\|\mathbf{e_I}\|_\infty$ are $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$ as expected
- $\|\mathbf{e_J}\|_\infty$ is much larger than when $\mathbf{e_J} \leftrightarrow \mathbf{e_I}$

# Solution-Discretization Error: $\varepsilon = \|\mathbf{e}\|_\infty$ (Discontinuity Removed)




- Discontinuity removed from $\mathbf{Z}$ using $\mathbf{C}$, corresponding MMS source term omitted in $\mathbf{V}^\mathcal{E}$:

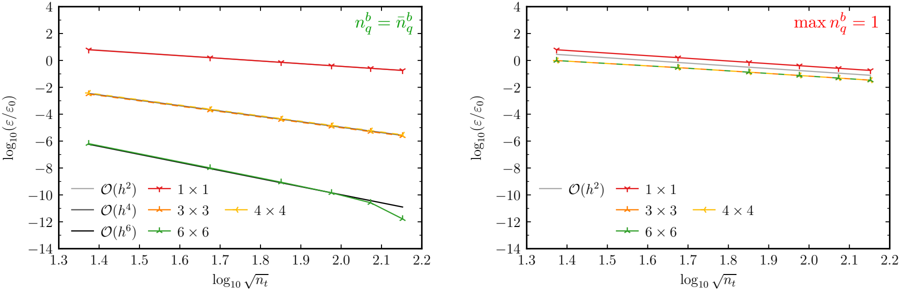
$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{Bmatrix}\mathbf{J}^h\\ \mathbf{I}^h\end{Bmatrix} = \begin{Bmatrix}\mathbf{V}^\mathcal{E}\\ \mathbf{0}\end{Bmatrix} \rightarrow \begin{bmatrix}\mathbf{A} & (\cancel{\mathbf{B}_1} + \mathbf{B}_2)\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{Bmatrix}\mathbf{J}^h\\ \mathbf{I}^h\end{Bmatrix} = \begin{Bmatrix}\mathbf{V}^\mathcal{E}\\ \mathbf{0}\end{Bmatrix}$$

- Convergence rates for $\|\mathbf{e_J}\|_\infty$ and $\|\mathbf{e_I}\|_\infty$ are $\mathcal{O}(h^2)$ as expected

- Correct implementation of $\mathbf{B}_1$ suggested by its removal using $\mathbf{C}$

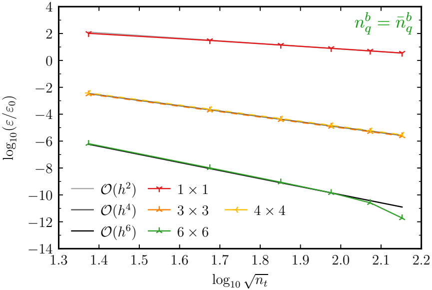
- Correct implementation of $\mathbf{C}$ suggested by expected convergence rates

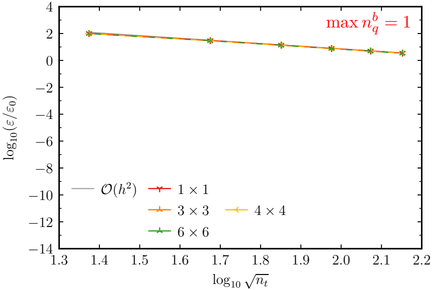
# Numerical-Integration Error: $\varepsilon = |e_a|$ $(G = G_2,\, d = d_1)$




- 2D points: [number for test integral] $\times$ [number for source integral]
- 1D points: $\bar{n}_q^b$ = number of 1D points with same convergence rate as 2D points
- Convergence rates are as expected for $n_q^b = \bar{n}_q^b$
- Convergence rates are limited to $\mathcal{O}(h^2)$ for $n_q^b = 1$

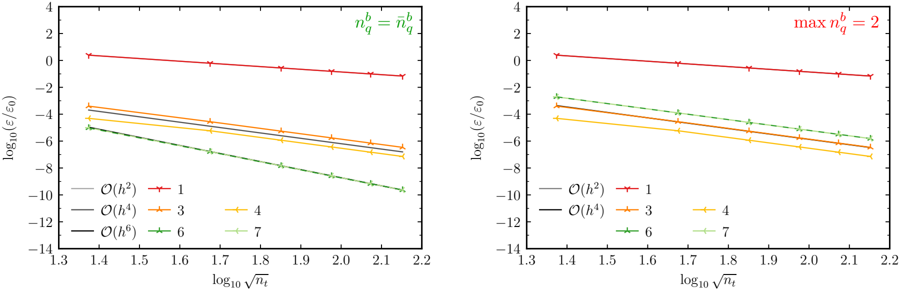
# Numerical-Integration Error: $\varepsilon = |e_a|$ ($G = G_2$, $d = d_3$)



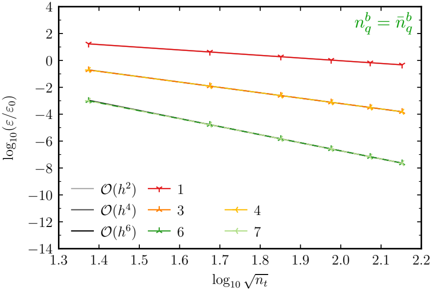

- 2D points: [number for test integral] $\times$ [number for source integral]
- 1D points: $\bar{n}_q^b$ = number of 1D points with same convergence rate as 2D points
- Convergence rates are as expected for $n_q^b = \bar{n}_q^b$
- Convergence rates are limited to $\mathcal{O}(h^2)$ for $n_q^b = 1$

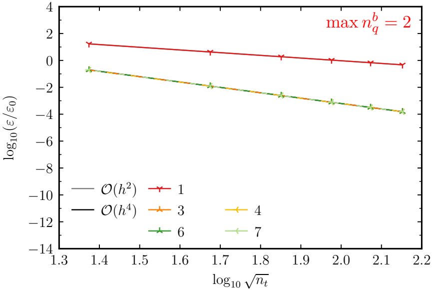
# Numerical-Integration Error: $\varepsilon = |e_b|$ ($G = G_2$, $d = d_1$)




- 2D points: number for test integral
- 1D points: $\bar{n}_q^b$ = number of 1D points with same convergence rate as 2D points
- Convergence rates are as expected for $n_q^b = \bar{n}_q^b$
- Convergence rates are limited to $\mathcal{O}(h^4)$ for $n_q^b = 2$

# Numerical-Integration Error: $\varepsilon = |e_b|$ $(G = G_2,\ d = d_3)$




- 2D points: number for test integral
- 1D points: $\bar{n}_q^b$ = number of 1D points with same convergence rate as 2D points
- Convergence rates are as expected for $n_q^b = \bar{n}_q^b$
- Convergence rates are limited to $\mathcal{O}(h^4)$ for $n_q^b = 2$

# Outline

- Introduction
- Governing Equations
- Code-Verification Approaches
- Numerical Examples
- Summary
  - Closing Remarks

# Closing Remarks

3 error sources in electromagnetic integral equations:

- **Domain-discretization error** – avoided
  - Considered planar surfaces
- **Solution-discretization error** – isolated
  - Manufactured $\mathbf{J}$, chose $\mathbf{I}_m$ to avoid source term
  - Manufactured Green's function (to integrate exactly)
  - Removed discontinuity to measure convergence rates without contamination
  - Demonstrated discontinuity implications by varying $\mathbf{e}_\mathbf{J} \leftrightarrow \mathbf{e}_\mathbf{I}$
- **Numerical-integration error** – isolated
  - Canceled basis-function contribution
  - Detected coding error

Achieved expected orders of accuracy

Questions? bafreno@sandia.gov brianfreno.github.io

# Additional Information


- B. Freno, N. Matula, W. Johnson
  Manufactured solutions for the method-of-moments implementation of the EFIE
  *Journal of Computational Physics* (2021) arXiv:2012.08681

- B. Freno, N. Matula, J. Owen, W. Johnson
  Code-verification techniques for the method-of-moments implementation of the EFIE
  *Journal of Computational Physics* (2022) arXiv:2106.13398

- B. Freno, N. Matula
  Code verification for practically singular equations
  *Journal of Computational Physics* (2022) arXiv:2204.01785

- B. Freno, N. Matula
  Code-verification techniques for the method-of-moments implementation of the MFIE
  *Journal of Computational Physics* (2023) arXiv:2209.09378

- B. Freno, N. Matula
  Code-verification techniques for the method-of-moments implementation of the CFIE
  *Journal of Computational Physics* (2023) arXiv:2302.06728

- B. Freno, N. Matula, R. Pfeiffer, E. Dohme, J. Kotulski
  Manufactured solutions for an electromagnetic slot model
  *Journal of Computational Physics* (2024) arXiv:2406.14573