Code-Verification Techniques for the Method-of-Moments Implementation of the Electric-Field Integral Equation

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Outline

- Introduction
- The Method of Moments Implementation of the EFIE
- Code-Verification Approach
- Numerical Examples
- Summary

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The Method of Moments Implementation of the EFIE

- Common technique for modeling electromagnetic scattering and radiation
- Surface of electromagnetic scatterer is discretized with elements
- 4D integrals are evaluated over 2D test and source elements
- Green's function yields singularities when test and source elements are near

Verification and Validation

Credibility of computational physics codes requires verification and validation

- **Validation** assesses how well models represent physical phenomena
  - Computational results are compared with experimental results
  - Assess suitability of models, model error, and bounds of validity

- **Verification** assesses accuracy of numerical solutions against expectations
  - *Solution verification* estimates numerical error for particular solution
  - *Code verification* verifies correctness of numerical-method implementation

Code Verification

- Code verification most rigorously assesses rate at which error decreases
- Requires exact solution, which is usually unavailable
- Manufactured solutions are popular alternative
  - Manufacture a solution
  - Insert manufactured solution into governing equations to get nonzero term
  - Add new term to equations to coerce solution to manufactured solution

Error Sources in the EFIE

3 sources of numerical error:

- **Domain discretization**: Representation of curved surfaces with planar elements
  - Second-order error for curved surfaces, no error for planar surfaces
  - Error reduced with curved elements – we focus on planar elements
- **Solution discretization**: Representation of solution or operators
  - Common in solution to differential, integral, and integro-differential equations
  - Finite number of basis functions to approximate solution
  - Finite samples queried to approximate underlying equation operators
- **Numerical integration**: Quadrature
  - Analytical integration is not always possible
  - For well-behaved integrands,
    - Expect integration error at least same order as solution-discretization error
    - Less rigorously, error should decrease with more quadrature points
  - For (nearly) singular integrands, monotonic convergence is not assured

This Work

Solution-discretization error

- Eliminate integration error by manufacturing solution and Green's function
- Equations become practically singular, permitting infinite solutions
- Reduce equations to constraints and find solution closest to manufactured

Numerical-integration error

- Cancel solution-discretization error
  $\rightarrow$ compute source term from basis-function representation of solution
- Eliminate solution-discretization error
  $\rightarrow$ avoid basis-function representation of solution and project onto solution

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The Electric-Field Integral Equation

In time-harmonic form, scattered electric field $\mathbf{E}^{\mathcal{S}}$ computed from surface current:

$$\mathbf{E}^{\mathcal{S}} = -(j\omega\mathbf{A} + \nabla\Phi)$$

Magnetic vector potential $\mathbf{A}(\mathbf{x}) = \mu \int_{S'} \mathbf{J}(\mathbf{x}')G_k(\mathbf{x},\mathbf{x}')dS'$

Electric scalar potential $\Phi(\mathbf{x}) = \frac{j}{\epsilon\omega} \int_{S'} \nabla' \cdot \mathbf{J}(\mathbf{x}')G_k(\mathbf{x},\mathbf{x}')dS'$ (Lorenz gauge condition)

$\mathbf{J}$ is surface current, $S' = S$ is surface of scatterer, μ and ϵ are permeability and permittivity of surrounding medium, and G_k is the Green's function

$$G_k(\mathbf{x},\mathbf{x}') = \frac{e^{-jkR}}{4\pi R},$$

where $R = |\mathbf{x} - \mathbf{x}'|$ and $k = \omega\sqrt{\mu\epsilon}$ is wave number

The Electric-Field Integral Equation (continued)

Total electric field $\mathbf{E} = \mathbf{E}^{\mathcal{I}} + \mathbf{E}^{\mathcal{S}}$

Incident electric field $\mathbf{E}^{\mathcal{I}}$ induces surface current $\mathbf{J}$

On surface S, tangential component of $\mathbf{E}$ is zero, such that

$$\mathbf{E}_t^{\mathcal{S}} = -\mathbf{E}_t^{\mathcal{I}}$$

$(\cdot)_t$ denotes tangential component

Compute $\mathbf{J}$ from $\mathbf{E}^{\mathcal{I}}$:

$$\mathbf{E}_t^{\mathcal{I}} = (j\omega\mathbf{A} + \nabla\Phi)_t$$

Variational Formulation

Project $\mathbf{E}_t^{\mathcal{I}} = (j\omega\mathbf{A} + \nabla\Phi)_t$ onto space $\mathbb{V}$ and integrate by parts

Space $\mathbb{V}$ contains vector fields
1) tangential to S
2) no components normal to boundary of S

Find $\mathbf{J} \in \mathbb{V}$, such that

$$a(\mathbf{J}, \mathbf{v}) = \left(\mathbf{E}^{\mathcal{I}}, \mathbf{v}\right) \quad \forall \mathbf{v} \in \mathbb{V},$$

where

$$\begin{aligned}
a(\mathbf{u}, \mathbf{v}) &= a^{\mathbf{A}}(\mathbf{u}, \mathbf{v}) + a^{\Phi}(\mathbf{u}, \mathbf{v}), \\
a^{\mathbf{A}}(\mathbf{u}, \mathbf{v}) &= j\omega\mu \int_S \bar{\mathbf{v}}(\mathbf{x}) \cdot \int_{S'} \mathbf{u}(\mathbf{x}') G_k(\mathbf{x}, \mathbf{x}') dS' dS, \\
a^{\Phi}(\mathbf{u}, \mathbf{v}) &= -\frac{j}{\epsilon\omega} \int_S \nabla \cdot \bar{\mathbf{v}}(\mathbf{x}) \int_{S'} \nabla' \cdot \mathbf{u}(\mathbf{x}') G_k(\mathbf{x}, \mathbf{x}') dS' dS, \\
(\mathbf{u}, \mathbf{v}) &= \int_S \mathbf{u}(\mathbf{x}) \cdot \bar{\mathbf{v}}(\mathbf{x}) dS
\end{aligned}$$

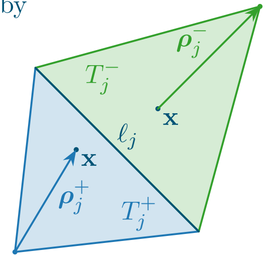
Rao–Wilton–Glisson Basis Functions

Discretize S with triangles and approximate $\mathbf{J}$ with basis-function representation:

$$\mathbf{J}_h(\mathbf{x}) = \sum_{j=1}^{n_b} J_j \boldsymbol{\Lambda}_j(\mathbf{x})$$

RWG basis functions defined for triangle pair by

$$\boldsymbol{\Lambda}_j(\mathbf{x}) = \begin{cases} \dfrac{\ell_j}{2A_j^+}\boldsymbol{\rho}_j^+, & \text{for } \mathbf{x} \in T_j^+ \\ \dfrac{\ell_j}{2A_j^-}\boldsymbol{\rho}_j^-, & \text{for } \mathbf{x} \in T_j^- \\ \mathbf{0}, & \text{otherwise} \end{cases}$$



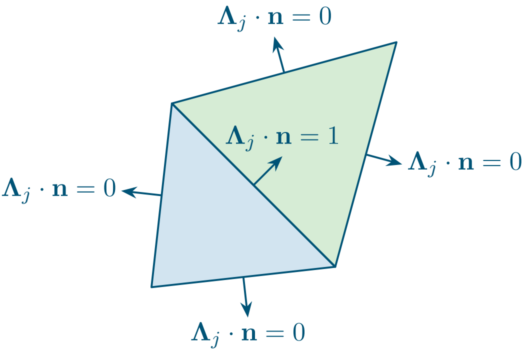
ℓ_j: length of edge

A_j^+ and A_j^-: areas of triangles T_j^+ and T_j^- associated with $\boldsymbol{\Lambda}_j$

$\boldsymbol{\rho}_j^+$: vector from vertex of T_j^+ opposite of shared edge to $\mathbf{x}$

$\boldsymbol{\rho}_j^-$: vector to vertex of T_j^- opposite of shared edge from $\mathbf{x}$

Rao–Wilton–Glisson Basis Functions (continued)



RWG basis functions ensure

- $\mathbf{J}_h$ is tangential to S
- $\mathbf{J}_h$ has no component normal to outer boundary of triangle pair

Along shared edge, component of $\mathbf{\Lambda}_j$ normal to edge is unity

- For edge shared by only 2 triangles, component of $\mathbf{J}_h$ normal to edge is J_j

Solution considered most accurate at edge midpoints

Discretized Problem

Find $\mathbf{J}_h \in \mathbb{V}_h$ (span of RWG basis functions), such that

$$a(\mathbf{J}_h, \mathbf{\Lambda}_i) = (\mathbf{E}^{\mathcal{I}}, \mathbf{\Lambda}_i)$$

for $i = 1, \dots, n_b$

In matrix–vector form, solve for $\mathbf{J}^h$:

$$\mathbf{Z}\mathbf{J}^h = \mathbf{V}$$

$$Z_{i,j} = a(\mathbf{\Lambda}_j, \mathbf{\Lambda}_i), \qquad J^h_j = J_j, \qquad V_i = (\mathbf{E}^{\mathcal{I}}, \mathbf{\Lambda}_i)$$

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  - Manufactured Surface Current and Green's Function
  - Solution-Discretization Error
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Manufactured Surface Current

Continuous equations: $r_i(\mathbf{J}\;\;) = a(\mathbf{J}\;\;, \mathbf{\Lambda}_i) - (\mathbf{E}^{\mathcal{I}}, \mathbf{\Lambda}_i) = 0$

Discretized equations: $r_{h_i}(\mathbf{J}_h) = a(\mathbf{J}_h, \mathbf{\Lambda}_i) - (\mathbf{E}^{\mathcal{I}}, \mathbf{\Lambda}_i) = 0$

Method of manufactured solutions modifies discretized equations:

$$\mathbf{r}_h(\mathbf{J}_h) = \mathbf{r}(\mathbf{J}_{\text{MS}}),$$

where $\mathbf{J}_{\text{MS}}$ is manufactured solution and $\mathbf{r}(\mathbf{J}_{\text{MS}})$ is computed exactly

Modified discretized equations: $a\left(\mathbf{J}_h, \mathbf{\Lambda}_i\right) = a\left(\mathbf{J}_{\text{MS}}, \mathbf{\Lambda}_i\right)$

Can be implemented via $\mathbf{E}^{\mathcal{I}}_{\text{MS}}$ if $(\mathbf{E}^{\mathcal{I}}_{\text{MS}}, \mathbf{\Lambda}_i) = a\left(\mathbf{J}_{\text{MS}}, \mathbf{\Lambda}_i\right) = V_{\text{MS}_i}$:

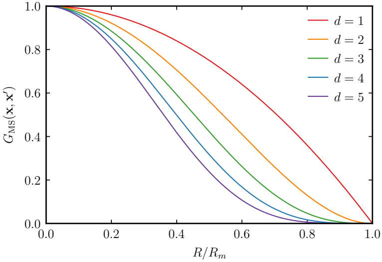
$$\mathbf{E}^{\mathcal{I}}_{\text{MS}}(\mathbf{x}) = j\omega\mathbf{A}(\mathbf{x}) + \nabla\Phi(\mathbf{x}) = \frac{j}{\omega\epsilon}\int_{S'}\left(k^2\mathbf{J}_{\text{MS}}(\mathbf{x}')G_k(\mathbf{x},\mathbf{x}') + \nabla'\cdot\mathbf{J}_{\text{MS}}(\mathbf{x}')\nabla G_k(\mathbf{x},\mathbf{x}')\right)dS'$$

Manufactured Green's Function

Integrals with G_k cannot be computed analytically or, when $R \to 0$, accurately

Inaccurately computing $\mathbf{E}^{\mathcal{I}}_{\text{MS}}(\mathbf{x})$ contaminates convergence studies

Manufacture Green's function: $G_{\text{MS}}(\mathbf{x},\mathbf{x}') = \left(1 - \frac{R^2}{R_m^2}\right)^d$, $R_m = \max\limits_{\mathbf{x},\mathbf{x}' \in S} R$ and $d \in \mathbb{N}$



Reasoning:

1) Even powers of R permit integrals to be computed analytically for many $\mathbf{J}_{\text{MS}}$
2) G_{MS} increases when R decreases, as with actual G_k

Solution-Discretization Error, G_{MS}: Optimization

G_{MS} makes $\mathbf{Z}$ practically singular $\rightarrow$ infinite solutions for $\mathbf{J}^h$

Choose $\mathbf{J}^h$ closest to $\mathbf{J}_n$ (J_{n_j}: $\mathbf{J}_{\text{MS}}$ from $T_j^+ \rightarrow T_j^-$) that satisfies $\mathbf{Z}\mathbf{J}^h = \mathbf{V}_{\text{MS}}$

Compute pivoted QR factorization of $\mathbf{Z}^H$:

$$\mathbf{Z}^H\mathbf{P} = [\mathbf{Q}_1,\, \mathbf{Q}_2]\begin{bmatrix}\mathbf{R}_1\\ \mathbf{0}\end{bmatrix} = \mathbf{Q}_1\mathbf{R}_1,$$

where $\mathbf{Z} \in \mathbb{C}^{n_b\times n_b}$, $\mathbf{Q}_1 \in \mathbb{C}^{n_b\times m_b}$, $\mathbf{Q}_2 \in \mathbb{C}^{n_b\times(n_b-m_b)}$, and $\mathbf{R}_1 \in \mathbb{C}^{m_b\times n_b}$

Numerically, pivoting facilitates determination of rank $m_b \leq n_b$ of $\mathbf{Z}$

Express $\mathbf{J}^h$ in terms of basis $\mathbf{Q}$:

$$\mathbf{J}^h = \mathbf{Q}_1\mathbf{u} + \mathbf{Q}_2\mathbf{v}$$

$\mathbf{u} \in \mathbb{C}^{m_b}$: coefficients that satisfy $\mathbf{Z}\mathbf{J}^h = \mathbf{V}_{\text{MS}}$
$\mathbf{v} \in \mathbb{C}^{n_b-m_b}$: coefficients that bring $\mathbf{J}^h$ closest to $\mathbf{J}_n$, given $\mathbf{u}$

Solution-Discretization Error, G_{MS}: Optimization (continued)

Constraints

$\mathbf{u}$: $\mathbf{R}_1^H\mathbf{u} = \mathbf{P}^T\mathbf{V}_{\text{MS}} \rightarrow \mathbf{J}^h$ satisfies $\mathbf{Z}\mathbf{J}^h = \mathbf{V}_{\text{MS}}$

Optimization

$\mathbf{v}$: $\arg\min_{\mathbf{v}} \left(\|\mathbf{e}_n\|_2^2 = \left(\mathbf{J}^h - \mathbf{J}_n\right)^H\left(\mathbf{J}^h - \mathbf{J}_n\right)\right) \rightarrow$ quadratic, $\mathbf{v} = \mathbf{Q}_2^H\mathbf{J}_n$

Result

$\mathbf{J}^h = \mathbf{J}_n + \mathbf{Q}_1\left(\mathbf{u} - \mathbf{Q}_1^H\mathbf{J}_n\right)$

Solution-Discretization Error, G_{MS}: Metric

For G_{MS}, measure L^∞-norm

$$\|\mathbf{e}_n\|_\infty = \max_j |e_{n_j}| \leq Ch^p$$

$\mathbf{e}_n = \mathbf{J}^h - \mathbf{J}_n$

J_{n_j}: component of $\mathbf{J}_{\text{MS}}$ flowing from T_j^+ to T_j^-

C : function of solution derivatives

h : measure of mesh size

p : order of accuracy

With multiple meshes, compute p from $\|\mathbf{e}_n\|_\infty$

For RWG basis functions, expectation is second-order accuracy ($p = 2$)

Solution-Discretization Error, G_k: Metric

For G_k, expectation is

$$\|\mathbf{e}\|_{H_{\text{div}}^{-1/2}(S)} \leq Ch^{3/2},$$

where $\mathbf{e}(\mathbf{x}) = \mathbf{J}_h(\mathbf{x}) - \mathbf{J}_{\text{MS}}(\mathbf{x})$ and

$$\|\mathbf{e}\|^2_{H_{\text{div}}^{-1/2}(S)} = \omega\mu \|\mathbf{e}\|^2_{H^{-1/2}(S)} + \frac{1}{\epsilon\omega} \|\nabla \cdot \mathbf{e}\|^2_{H^{-1/2}(S)},$$

$$\|\mathbf{e}\|^2_{H^{-1/2}(S)} = \int_S \bar{\mathbf{e}}(\mathbf{x}) \cdot \int_{S'} \mathbf{e}(\mathbf{x}') G_0(\mathbf{x}, \mathbf{x}') dS' dS$$

Numerical-integration error is incurred when computing $\mathbf{Z}$ and the norm

Usefulness of this alone for code verification is limited

G_{MS} avoids contamination from numerical-integration error

This can confirm the singularities are integrated suitably

Numerical-Integration Error: Solution-Discretization Error Cancellation

Cancellation introduces basis functions on both sides:

$$\begin{aligned} a\left(\mathbf{J}_h, \boldsymbol{\Lambda}_i\right) &= a\left(\mathbf{J}_{h_{\text{MS}}}, \boldsymbol{\Lambda}_i\right) \\ &= \left(\mathbf{E}^{\mathcal{I}}_{h_{\text{MS}}}, \boldsymbol{\Lambda}_i\right) \end{aligned}$$

$\mathbf{J}_{h_{\text{MS}}}$ is basis-function representation of $\mathbf{J}_{\text{MS}}$

$\mathbf{E}^{\mathcal{I}}_{h_{\text{MS}}}$ obtained from $\mathbf{J}_{h_{\text{MS}}}$ instead of $\mathbf{J}_{\text{MS}}$

Replace $a\left(\mathbf{J}_h, \boldsymbol{\Lambda}_i\right)$ with quadrature approximation $a_h\left(\mathbf{J}_h, \boldsymbol{\Lambda}_i\right)$

Compute $\left(\mathbf{E}^{\mathcal{I}}_{h_{\text{MS}}}, \boldsymbol{\Lambda}_i\right)$ exactly for G_{MS} or with sufficient accuracy for G_k

Measure $\|\mathbf{e}_n\|_\infty = \max_j |e_{n_j}|$

Numerical-Integration Error: Solution-Discretization Error Elimination

Cancellation assesses numerical integration with limited-order basis functions

Elimination removes basis functions:

$$a(\mathbf{J}_{\text{MS}}, \mathbf{J}_{\text{MS}}) = (\mathbf{E}^{\mathcal{I}}_{\text{MS}}, \mathbf{J}_{\text{MS}})$$

Replace $a(\mathbf{J}_{\text{MS}}, \mathbf{J}_{\text{MS}})$ with quadrature approximation $a_h(\mathbf{J}_{\text{MS}}, \mathbf{J}_{\text{MS}})$

Measure relative error $|I_h - I|/|I|$, where $I = a(\mathbf{J}_{\text{MS}}, \mathbf{J}_{\text{MS}})$ and $I_h = a_h(\mathbf{J}_{\text{MS}}, \mathbf{J}_{\text{MS}})$

Compute I_h from quadrature integration over triangular discretization

Compute I exactly for G_{MS} or with sufficient accuracy for G_k

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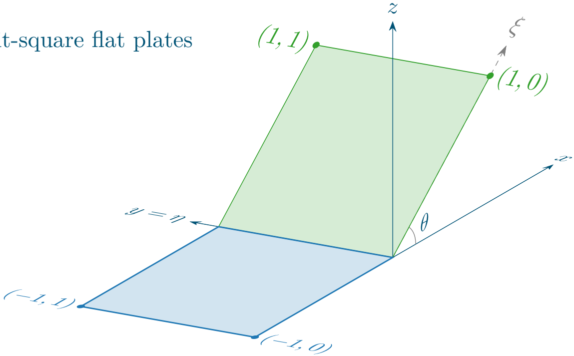
Overview

Surface S: 2 unit-square flat plates

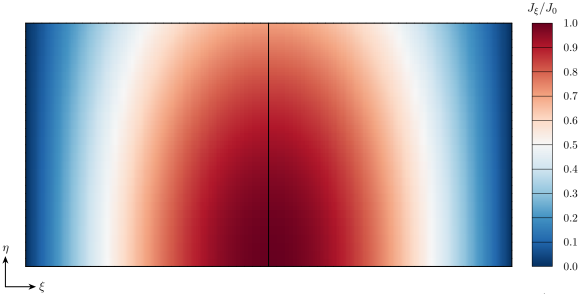


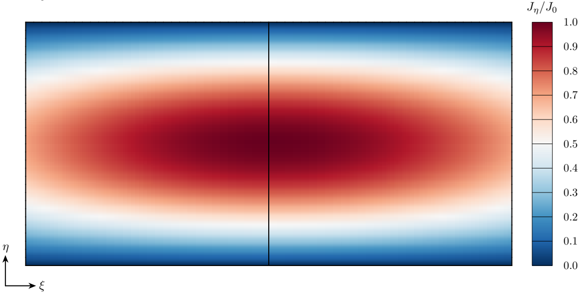
Manufactured surface current $\mathbf{J}_{\text{MS}}(\mathbf{x}) = \begin{Bmatrix} J_\xi(\boldsymbol{\xi}) = J_0\cos(\pi\xi/2)\cos(\pi\eta/4) \\ J_\eta(\boldsymbol{\xi}) = J_0\cos(\pi\xi/4)\sin(\pi\eta) \end{Bmatrix}$

Plate-fixed coordinate system $\boldsymbol{\xi}(\mathbf{x};\theta) = \begin{Bmatrix} \xi(\mathbf{x};\theta) = \frac{1}{L_0}\begin{cases} x, & \text{for } x \le 0 \text{ m} \\ x\cos\theta + z\sin\theta, & \text{for } x > 0 \text{ m} \end{cases} \\ \eta(\mathbf{x}) = y/L_0, \end{Bmatrix}$

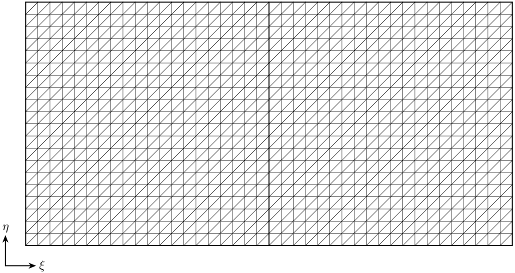
$J_0 = 1$ A/m and $L_0 = 1$ m

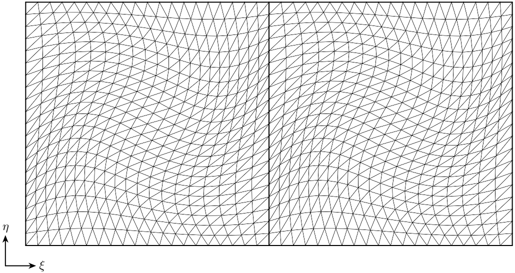
Manufactured Surface Current $\mathbf{J}_{\mathrm{MS}}$





Uniform and Twisted Meshes, with $n_t = 1600$





Potential Contributions

Account for disparities in magnitudes of contributions to $\mathbf{Z}$ from $\mathbf{A}$ and Φ:

$$\mathbf{Z} = \mathbf{Z}^{\mathbf{A}} + \mathbf{Z}^{\Phi},$$

where

$$Z_{i,j}^{\mathbf{A}} = a^{\mathbf{A}}(\mathbf{\Lambda}_j, \mathbf{\Lambda}_i),$$
$$Z_{i,j}^{\Phi} = a^{\Phi}(\mathbf{\Lambda}_j, \mathbf{\Lambda}_i)$$

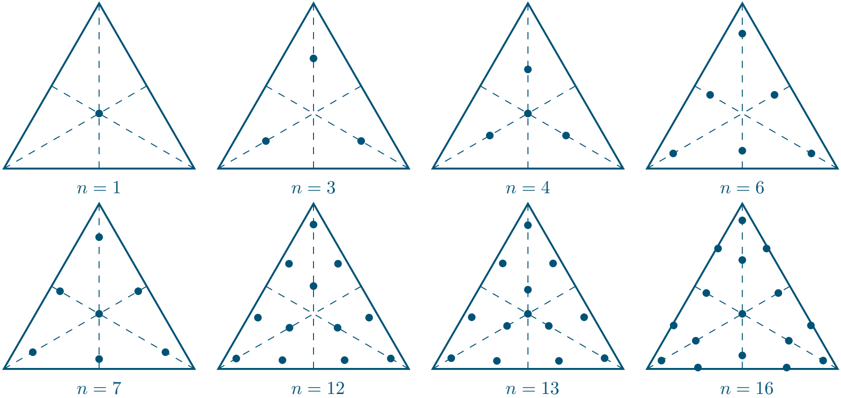
Consider $\mathbf{Z}^{\mathbf{A}}$ and $\mathbf{Z}^{\Phi}$ together and separately, with $\epsilon = 1$ F/m and $\mu = 1$ H/m

Together, set $k = 1$ m^{-1} for $\mathbf{Z}$

Separately, for G_{MS}, $\lim_{k\to\infty} \mathbf{Z} = \mathbf{Z}^{\mathbf{A}}$ and $\lim_{k\to 0} \mathbf{Z} = \mathbf{Z}^{\Phi}$

Similarly with $I = I^{\mathbf{A}} + I^{\Phi}$, where $I^{\mathbf{A}} = a^{\mathbf{A}}(\mathbf{J}_{\text{MS}}, \mathbf{J}_{\text{MS}})$ and $I^{\Phi} = a^{\Phi}(\mathbf{J}_{\text{MS}}, \mathbf{J}_{\text{MS}})$

Polynomial Quadrature Rules



n	1	3	4	6	7	12	13	16
Max. integrand degree	1	2	3	4	5	6	7	8
Convergence rate	$\mathcal{O}(h^2)$	$\mathcal{O}(h^4)$	$\mathcal{O}(h^4)$	$\mathcal{O}(h^6)$	$\mathcal{O}(h^6)$	$\mathcal{O}(h^8)$	$\mathcal{O}(h^8)$	$\mathcal{O}(h^{10})$

Manufactured Green's Function G_{MS}

Consider $d = 1$ and $d = 2$ for $G_{\text{MS}} = \left(1 - \frac{R^2}{R_m^2}\right)^d$

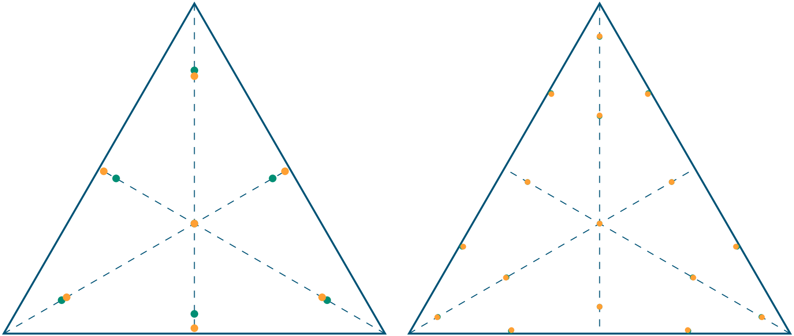
For $d = 1$,

- $\mathbf{E}_{\text{MS}}^{\mathcal{I}}$ is polynomial of degree 2
- Integrand of $\mathbf{V}_{\text{MS}}$ is polynomial of degree 3
- $\mathbf{V}_{\text{MS}}$ integrated exactly with 4 polynomial quadrature points

For $d = 2$,

- $\mathbf{E}_{\text{MS}}^{\mathcal{I}}$ is polynomial of degree 4
- Integrand of $\mathbf{V}_{\text{MS}}$ is polynomial of degree 5
- $\mathbf{V}_{\text{MS}}$ integrated exactly with 7 polynomial quadrature points

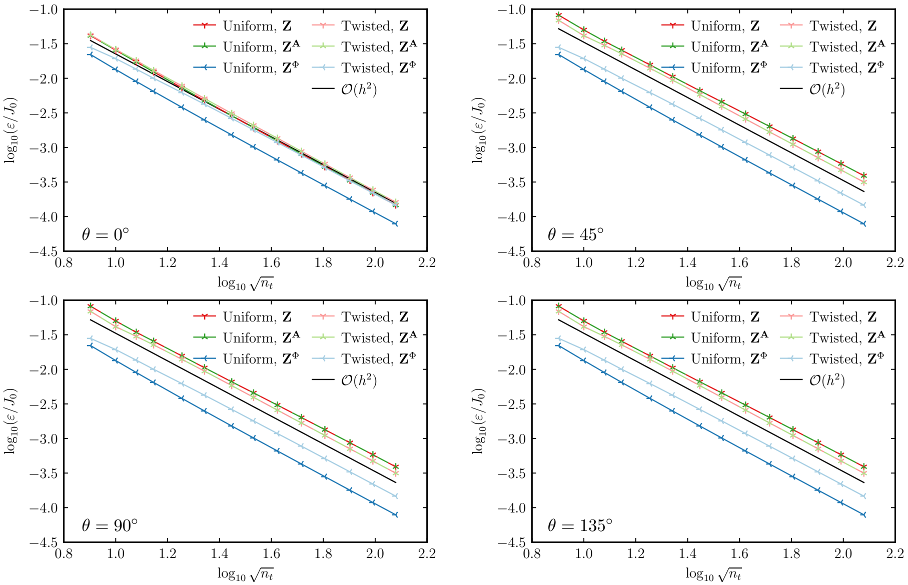
Actual Green's Function G_k



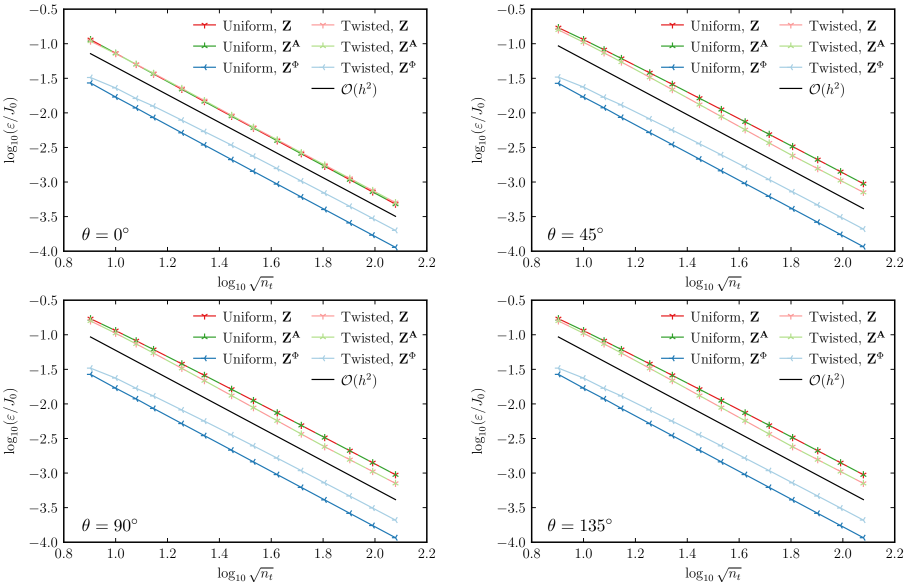
$n = 7$ $n = 16$

	Combination	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
Test Points	Polynomial Rules	7	7	16	16	7	7	16	16
	(Near-)Singular	—	—	—	—	7	7	16	16
Source Points	Polynomial Rules	7	16	7	16	7	16	7	16
	Radial	3	6	3	6	3	6	3	6
	Transverse	12	24	12	24	12	24	12	24

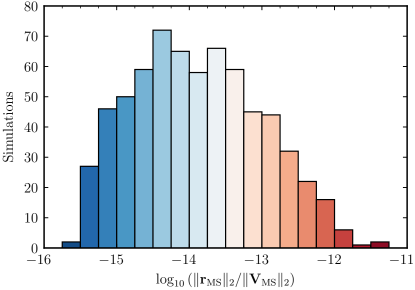
Discretization Error, G_{MS}: $\varepsilon = \|\mathbf{e}_n\|_\infty$ $(d = 1)$



Discretization Error, G_{MS}: $\varepsilon = \|\mathbf{e}_n\|_\infty$ $(d = 2)$



Discretization Error, G_{MS}: Residual $\mathbf{r}_{\text{MS}} = \mathbf{Z}\mathbf{J}^h - \mathbf{V}_{\text{MS}}$



All residuals are less than 10^{-11}

Discretization Error, G_{MS}: Maximum Rank across Meshes $\max m_b$

	Uniform			Twisted		
θ	$\mathbf{Z}$	$\mathbf{Z}^{\mathbf{A}}$	$\mathbf{Z}^{\Phi}$	$\mathbf{Z}$	$\mathbf{Z}^{\mathbf{A}}$	$\mathbf{Z}^{\Phi}$
0°	8	8	2	8	8	2
45°	13	13	3	13	13	3
90°	13	13	3	13	13	3
135°	13	13	3	13	13	3

$d = 1$

	Uniform			Twisted		
θ	$\mathbf{Z}$	$\mathbf{Z}^{\mathbf{A}}$	$\mathbf{Z}^{\Phi}$	$\mathbf{Z}$	$\mathbf{Z}^{\mathbf{A}}$	$\mathbf{Z}^{\Phi}$
0°	18	18	7	18	18	7
45°	31	31	11	31	31	11
90°	31	31	11	31	31	11
135°	31	31	11	31	31	11

$d = 2$

Discretization Error, G_{MS}: Coding Errors

Given low rank of constraints, can coding errors be detected?

Consider 4 coding errors:

Case 1: *Incorrect value of k.* k is increased by 1%
$\rightarrow$ incorrectly weighted contribution to $\mathbf{Z}$ from $\mathbf{Z}_{\mathbf{A}}$

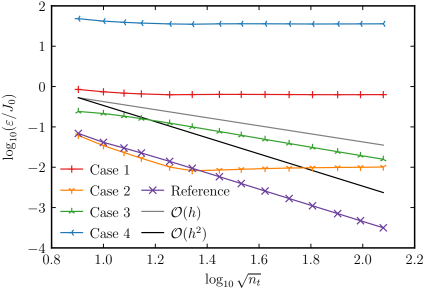
Case 2: *Incorrect quadrature weights.* Weights are increased by 1%
$\rightarrow$ inconsistent solutions to integrals

Case 3: *Incorrect matrix entry.* $Z_{1,2}$ is increased by 1%

Case 4: *Incorrect triangle areas.* Areas from uniform mesh used in basis function computations instead of actual areas

Twisted mesh, both contributions to $\mathbf{Z}$, $d = 1$, and $\theta = 45°$

Discretization Error, G_{MS}: Coding Errors (continued)



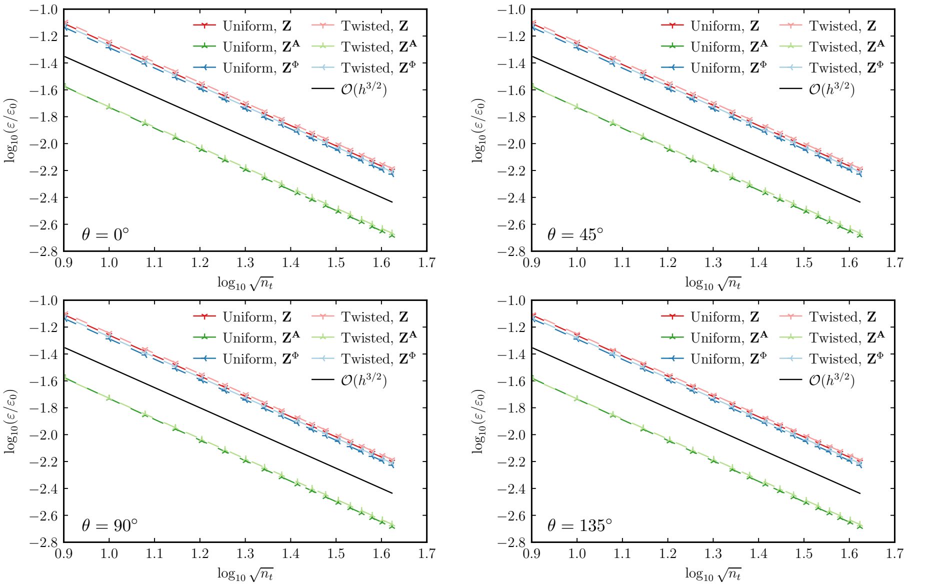
Case 1: *Incorrect value of* $k \to \mathcal{O}(1)$

Case 2: *Incorrect quadrature weights* $\to \mathcal{O}(1)$

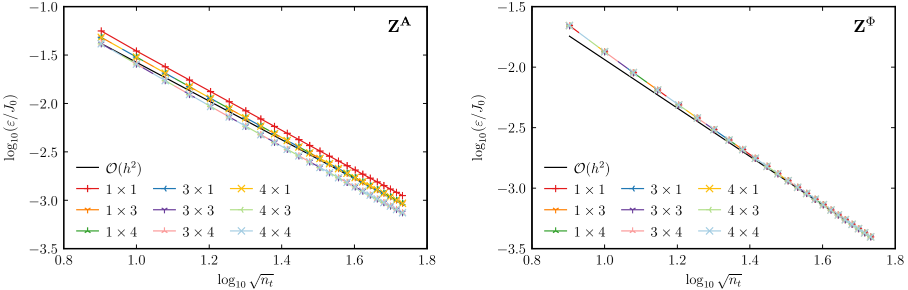
Case 3: *Incorrect matrix entry* $\to \mathcal{O}(h)$

Case 4: *Incorrect triangle areas* $\to \mathcal{O}(1)$

Discretization Error, G_k: $\varepsilon = \|\mathbf{e}\|_{H_{\mathrm{div}}^{-1/2}(S)}$ (Q8)

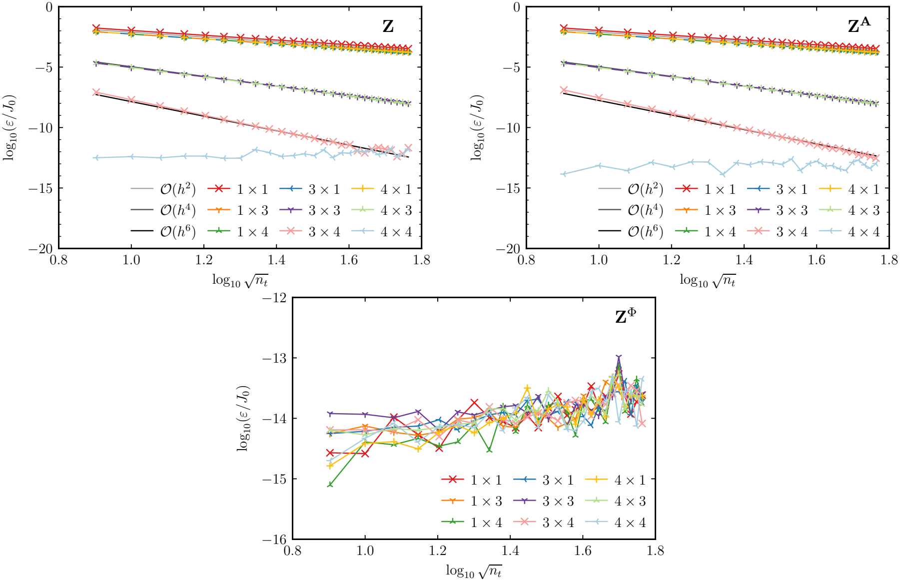


Integration Error, G_{MS}: $\varepsilon = \|\mathbf{e}_n\|_\infty$ $(d = 1,\ \theta = 0^\circ)$

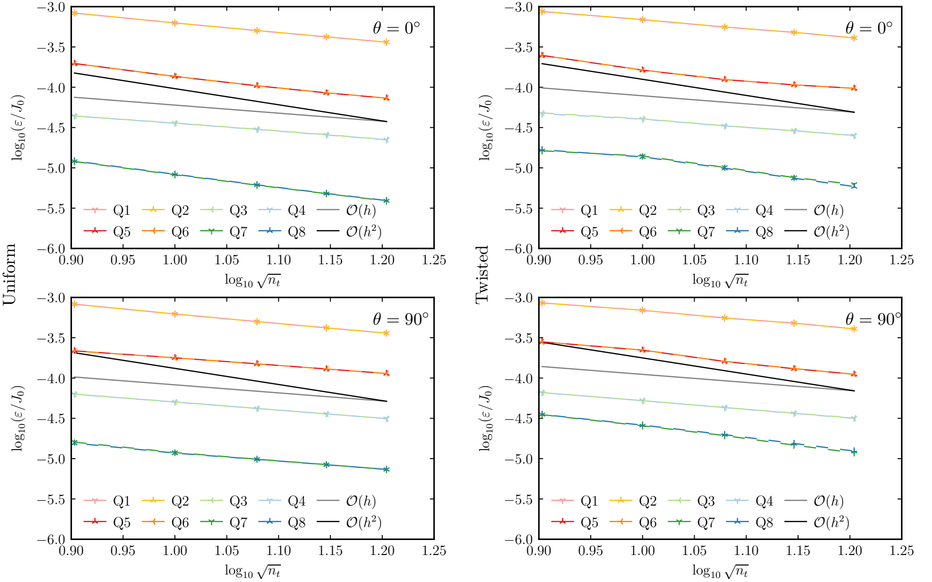


Solution-discretization error is $\mathcal{O}(h^2)$
→ contaminates numerical-integration error studies

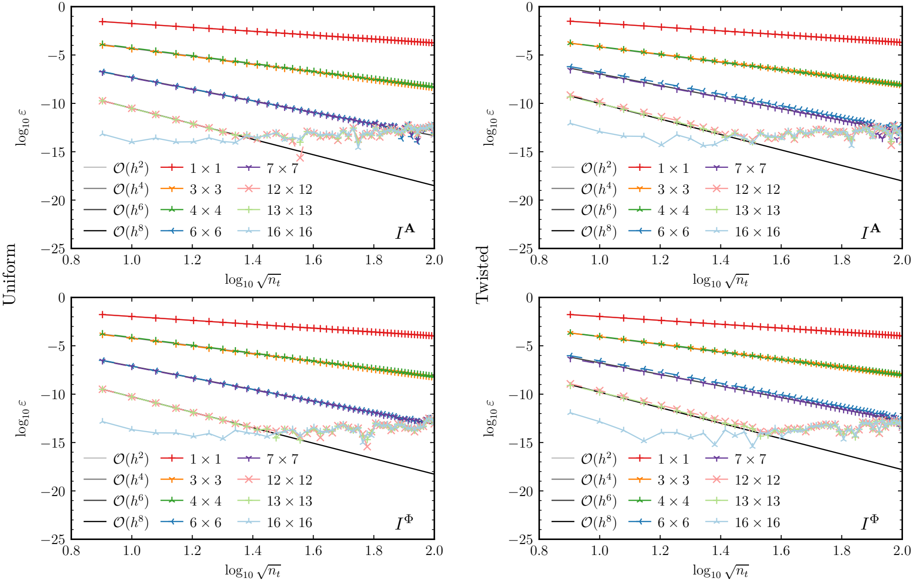
Integration Error, Cancellation, G_{MS}: $\varepsilon = \|\mathbf{e}_n\|_\infty$ $(d = 1,\ \theta = 0°)$



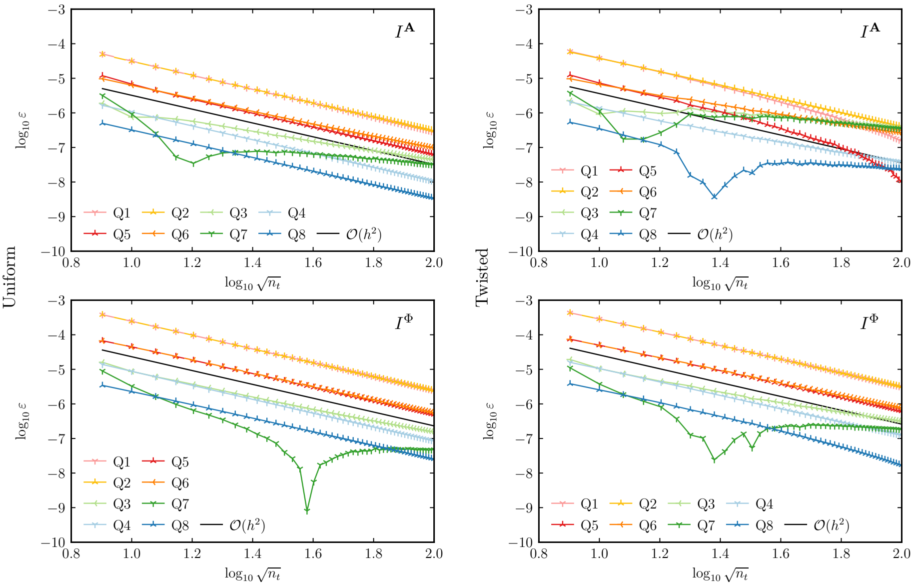
Integration Error, Cancellation, G_k: $\varepsilon = \|\mathbf{e}_n\|_\infty$



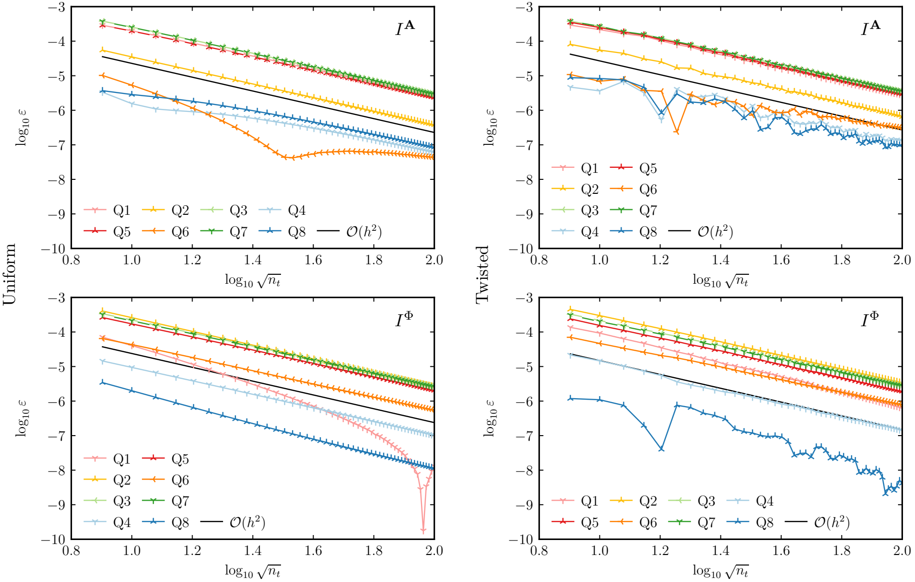
Integration Error, Elimination, G_{MS}: $\varepsilon = |I_h - I|/|I|$ ($d = 1$, $\theta = 0°$)



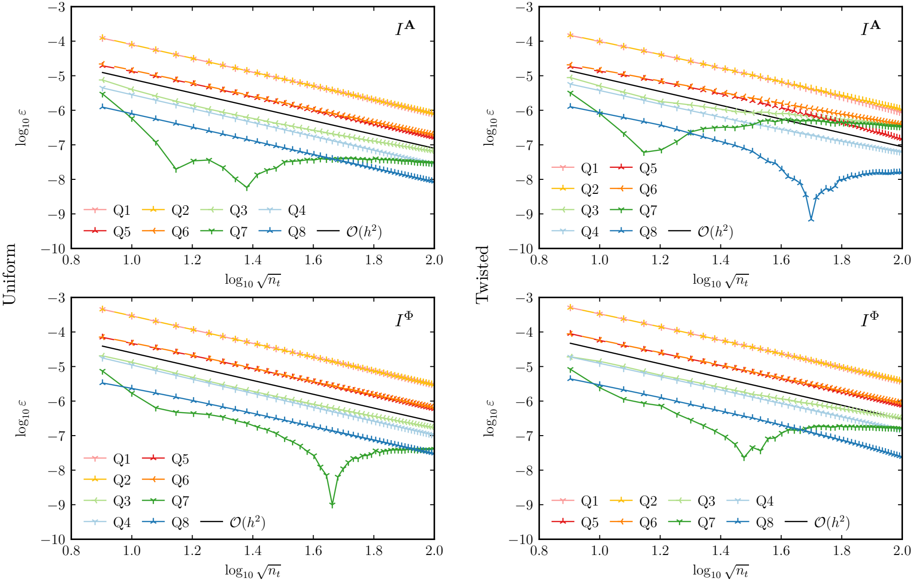
Integration Error, Elimination, G_k: $\varepsilon = |I_h - I|/|I|$ ($\theta = 0°$)



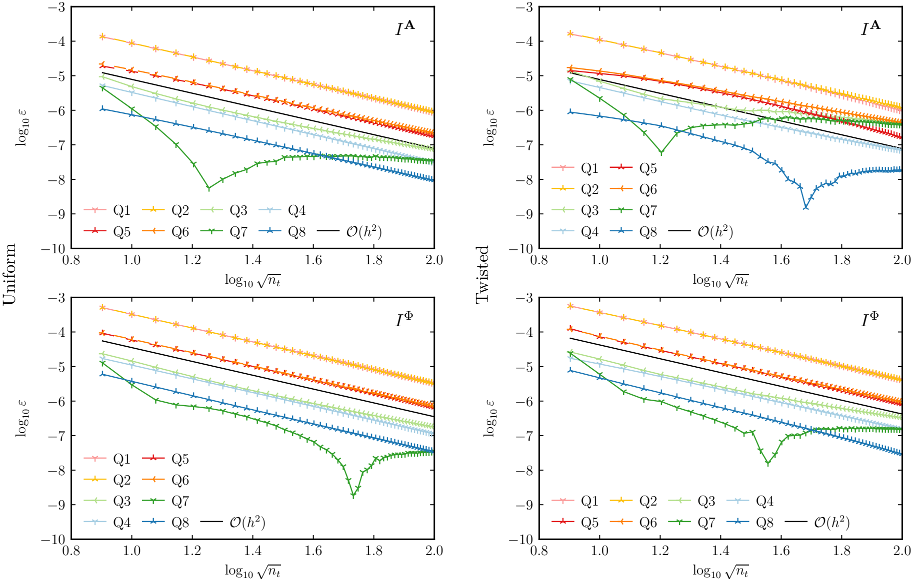
Integration Error, Elimination, G_k: $\varepsilon = |I_h - I|/|I|$ ($\theta = 45°$)



Integration Error, Elimination, G_k: $\varepsilon = |I_h - I|/|I|$ ($\theta = 90°$)



Integration Error, Elimination, G_k: $\varepsilon = |I_h - I|/|I|$ ($\theta = 135°$)



Outline

- Introduction
- The Method of Moments Implementation of the EFIE
- Code-Verification Approach
- Numerical Examples
- Summary
  - Closing Remarks

Closing Remarks

- Several combinations of error sources in MoM implementation of EFIE
  - Important to isolate and measure errors
- Solution-discretization error
  - Manufacture both the surface current and Green's function
  - Reduce the governing equations to a constraint
  - Optimize to find solution closest to manufactured solution
  - Approach is effective for properly and improperly coded examples
- Numerical-integration error
  - Cancel solution-discretization error
    $\rightarrow$ compute source term from basis-function representation of solution
  - Eliminate solution-discretization error
    $\rightarrow$ avoid basis-function representation of solution and project onto solution
  - Quadrature rules do not monotonically converge for singular integrands

Additional Information

- B. Freno, N. Matula, W. Johnson
  Manufactured solutions for the method-of-moments implementation of the electric-field integral equation
  Journal of Computational Physics (2021) arXiv:2012.08681

- B. Freno, N. Matula, J. Owen, W. Johnson
  Code-verification techniques for the method-of-moments implementation of the electric-field integral equation
  Journal of Computational Physics (2022) arXiv:2106.13398

- B. Freno, N. Matula
  Code verification for practically singular equations
  arXiv:2204.01785

- B. Freno, W. Johnson, B. Zinser, S. Campione
  Symmetric triangle quadrature rules for arbitrary functions
  Computers & Mathematics with Applications (2020) arXiv:1909.01480

- B. Freno, W. Johnson, B. Zinser, D. Wilton, F. Vipiana, S. Campione
  Characterization and integration of the singular test integrals in the method-of-moments implementation of the EFIE
  Engineering Analysis with Boundary Elements (2021) arXiv:1911.02107

Questions?

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