# Code-Verification Techniques for an Arbitrary-Depth Electromagnetic Slot Model

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# Outline

- Introduction
- Governing Equations
- Code-Verification Approaches
- Numerical Examples
- Summary

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- Introduction
  - Electromagnetic Integral Equations
  - Verification and Validation
  - Error Sources
  - This Work
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# Electromagnetic Integral Equations

- Are commonly used to model electromagnetic scattering and radiation
- Relate surface current to incident electric and/or magnetic field
- Discretize surface of electromagnetic scatterer with elements
- Evaluate 4D reaction integrals over 2D test and source elements
- Contain singular integrands when test and source elements are near

# Electromagnetic Aperture and Slot Models

- EM penetration occurs through openings of otherwise closed surfaces
- Penetration may occur intentionally or unintentionally
- Slot connects exterior surface of scatterer to interior surface of cavity
- Model slot as wires carrying magnetic current located at apertures
  - Exterior surface interacts with exterior wire
  - Interior surface interacts with interior wire
  - Exterior and interior wires interact with each other
  - Exterior and interior surfaces do not interact directly

# Verification and Validation

Credibility of computational physics codes requires verification and validation

- **Validation** assesses how well models represent physical phenomena
  - Compare computational results with experimental results
  - Assess suitability of models, model error, and bounds of validity
- **Verification** assesses accuracy of numerical solutions against expectations
  - *Solution verification* estimates numerical error for particular solution
  - *Code verification* verifies correctness of numerical-method implementation

# Code Verification

- Code verification most rigorously assesses rate at which error decreases
- Error requires exact solution – usually unavailable
- Manufactured solutions are popular alternative
  - Manufacture an arbitrary solution
  - Insert manufactured solution into governing equations to get residual term
  - Add residual term to equations to coerce solution to manufactured solution
- For integral equations, few instances of code verification exist
- Analytical differentiation is straightforward – analytical integration is not
- Numerical integration is necessary, generally incurs an approximation error
- Therefore, manufactured source term may have its own numerical error

# Error Sources in the Electromagnetic Integral Equations

3 sources of numerical error:

- **Domain discretization**: Representation of curved surfaces with planar elements
  - Second-order error for curved surfaces, no error for planar surfaces
  - Error reduced with curved elements
- **Solution discretization**: Representation of solution or operators
  - Common in solution to differential, integral, and integro-differential equations
  - Finite number of basis functions to approximate solution
  - Finite samples queried to approximate underlying equation operators
- **Numerical integration**: Quadrature
  - Analytical integration is not always possible
  - For well-behaved integrands,
    - Expect integration error at least same order as solution-discretization error
    - Less rigorously, error should decrease with more quadrature points
  - For (nearly) singular integrands, monotonic convergence is not assured

# This Work

**Isolate solution-discretization error**

- Manufacture solution
- Eliminate numerical-integration error by manufacturing Green's function
- Mitigate contamination from discontinuity due to wire–surface interaction

**Isolate numerical-integration error**

- Manufacture solution
- Cancel solution-discretization error using basis functions

**Avoid domain-discretization error**

- Consider only planar surfaces
- Previously provided approaches to account for domain-discretization error

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  - The Electric-Field Integral Equation
  - The Slot Equation
  - Discretization
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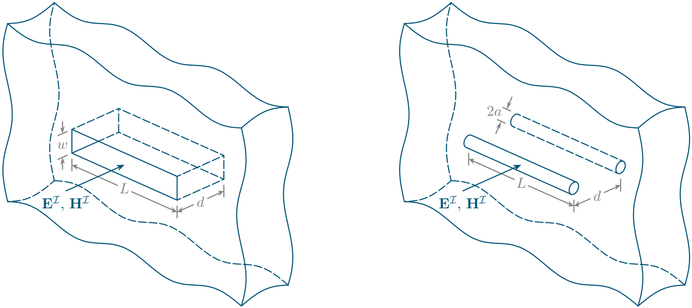
# Arbitrary-Depth Slot Model Overview




- Electromagnetic scatterer of arbitrary depth $d$ encloses a cavity
- Exterior is connected to interior by rectangularly prismatic slot with $L \gg w$ (left)
- Slot is replaced with two thin wires at apertures that carry magnetic current (right)
- Exterior and interior surfaces interact with wires, not each other
- Wires interact with each other through waveguide model
- EFIE solved on each surface, slot equation solved for wires

# The Electric-Field Integral Equation

In time-harmonic form, $\mathbf{E}^{\mathcal{S}}$ computed from $\mathbf{J}$ and $\mathbf{M}$

Scattered electric field $$\mathbf{E}^{\mathcal{S}}(\mathbf{x}) = -\left(j\omega\mathbf{A}(\mathbf{x}) + \nabla\Phi(\mathbf{x}) + \frac{1}{\epsilon}\nabla\times\mathbf{F}(\mathbf{x})\right)$$

Magnetic vector potential $$\mathbf{A}(\mathbf{x}) = \mu\int_{S'}\mathbf{J}(\mathbf{x}')G(\mathbf{x},\mathbf{x}')dS'$$

Electric scalar potential $$\Phi(\mathbf{x}) = \frac{j}{\epsilon\omega}\int_{S'}\nabla'\cdot\mathbf{J}(\mathbf{x}')G(\mathbf{x},\mathbf{x}')dS'$$

Electric vector potential $$\mathbf{F}(\mathbf{x}) = \epsilon\int_{S'}\mathbf{M}(\mathbf{x}')G(\mathbf{x},\mathbf{x}')dS'$$

Green's function $$G(\mathbf{x},\mathbf{x}') = \frac{e^{-jkR}}{4\pi R}, \qquad R = |\mathbf{x}-\mathbf{x}'|$$

Singularity when $R \to 0$

$\mathbf{J}$ and $\mathbf{M}$ are electric and magnetic surface current densities
$S' = S$ is surface of scatterer
$\mu$ and $\epsilon$ are permeability and permittivity of surrounding medium
$k = \omega\sqrt{\mu\epsilon}$ is wavenumber

# The Electric-Field Integral Equation (continued)

Compute $\mathbf{J}$ and $\mathbf{M}$ from incident electric field $\mathbf{E}^{\mathcal{I}}$ $\quad (\mathbf{n} \times (\mathbf{E}^{\mathcal{S}} + \mathbf{E}^{\mathcal{I}}) = Z_s \mathbf{n} \times \mathbf{J})$

Discretize surface with triangles, approximate $\mathbf{J}$ with RWG basis functions:

$$\mathbf{J}_h(\mathbf{x}) = \sum_{j=1}^{n_b} J_j \boldsymbol{\Lambda}_j(\mathbf{x})$$

Project EFIE onto linear, vector-valued RWG basis functions

Express $\mathbf{M}$ in terms of filament magnetic current $\mathbf{I}_m$

Discretize wire with bars, approximate $\mathbf{I}_m$ with 1D basis functions:

$$\mathbf{I}_h(s) = \sum_{j=1}^{n_b^m} I_j \boldsymbol{\Lambda}_j^m(s)$$

# The Slot Equation

The magnetic current along the slot is modeled using a waveguide model:

$$\mathbf{s}\cdot(\mathbf{J}^{\pm}\times\mathbf{n}^{\pm}) + \frac{j\omega\epsilon}{2wL(k^2-\beta_x^2)}\sum_{p=1}^{\infty}\beta_{y_p}\int_0^L \sin\left(\frac{p\pi s}{L}\right)\sin\left(\frac{p\pi s'}{L}\right)\times$$

$$\left(\pm\left[I_m^-(s')-I_m^+(s')\right]\tan(\beta_{y_p}d/2)+\left[I_m^+(s')+I_m^-(s')\right]\cot(\beta_{y_p}d/2)\right)ds' = 0$$

$I_m(0) = I_m(L) = 0$

$\mathbf{s}$ is the direction of the wire ($\mathbf{I}_m = I_m(s)\mathbf{s}$)

Superscripts $-$ and $+$ denote exterior and interior

Effective wire radius $a$ obtained from conformal mapping using $w$ and $d$

$\beta_\alpha$ is the propagation constant in the $\alpha$ direction

Project slot equation onto 1D basis functions

# Discretized Problem

Find $\mathbf{J}_h \in \mathbb{V}_h$ and $\mathbf{I}_h \in \mathbb{V}_h^m$, such that

$$a_{\mathcal{E},\mathcal{E}}(\mathbf{J}_h, \mathbf{\Lambda}_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_h, \mathbf{\Lambda}_i) = b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}}, \mathbf{\Lambda}_i) \qquad \text{for } i = 1, \dots, n_b \quad \text{(EFIE)}$$

$$a_{\mathcal{M},\mathcal{E}}(\mathbf{J}_h, \mathbf{\Lambda}_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_h, \mathbf{\Lambda}_i^m) = 0 \qquad \text{for } i = 1, \dots, n_b^m \quad \text{(Slot)}$$

Evaluate EFIE on exterior and interior surfaces: $n_b^- + n_b^+$ unknowns for $\mathbf{J}_h$

Evaluate slot equation on exterior and interior wires: $n_b^{m-} + n_b^{m+}$ unknowns for $\mathbf{I}_h$

# Matrix–Vector Form

In matrix–vector form, solve for $\boldsymbol{\mathcal{J}}^h$:

$$\mathbf{Z}\boldsymbol{\mathcal{J}}^h = \mathbf{V}$$

$$\mathbf{Z} = \underbrace{\begin{bmatrix} \mathbf{A}^- & 0 & \mathbf{B}^- & 0 \\ 0 & \mathbf{A}^+ & 0 & \mathbf{B}^+ \\ \mathbf{C}^- & 0 & \mathbf{D}^-_{\sim} & \mathbf{D}^-_{\not\sim} \\ 0 & \mathbf{C}^+ & \mathbf{D}^+_{\not\sim} & \mathbf{D}^+_{\sim} \end{bmatrix}}_{\text{Impedance matrix}}, \qquad \boldsymbol{\mathcal{J}}^h = \underbrace{\begin{Bmatrix} \mathbf{J}^{h-} \\ \mathbf{J}^{h+} \\ \mathbf{I}^{h-} \\ \mathbf{I}^{h+} \end{Bmatrix}}_{\text{Current vector}}, \qquad \mathbf{V} = \underbrace{\begin{Bmatrix} \mathbf{V}^{\mathcal{E}-} \\ \mathbf{V}^{\mathcal{E}+} \\ 0 \\ 0 \end{Bmatrix}}_{\text{Excitation vector}},$$

where

$$A_{i,j} = a_{\mathcal{E},\mathcal{E}}(\boldsymbol{\Lambda}_j, \boldsymbol{\Lambda}_i), \quad B_{i,j} = a_{\mathcal{E},\mathcal{M}}(\boldsymbol{\Lambda}_j^m, \boldsymbol{\Lambda}_i), \quad C_{i,j} = a_{\mathcal{M},\mathcal{E}}(\boldsymbol{\Lambda}_j, \boldsymbol{\Lambda}_i^m), \quad D_{i,j} = a_{\mathcal{M},\mathcal{M}}(\boldsymbol{\Lambda}_j^m, \boldsymbol{\Lambda}_i^m),$$

$$J_j^h = J_j, \qquad I_j^h = I_j, \qquad V_j^{\mathcal{E}} = b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}}, \boldsymbol{\Lambda}_i)$$

More compactly: $\mathbf{Z} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \quad \boldsymbol{\mathcal{J}}^h = \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix}, \quad \mathbf{V} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}} \\ 0 \end{Bmatrix}$

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  - Solution-Discretization Error
  - Numerical-Integration Error
  - Manufactured Green's Function
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# Manufactured Solutions for the EFIE

Continuous: $r_{\mathcal{E}_i}(\mathbf{J}\ ,\mathbf{I}_m) = a_{\mathcal{E},\mathcal{E}}(\mathbf{J}\ ,\boldsymbol{\Lambda}_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_m,\boldsymbol{\Lambda}_i) - b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}},\boldsymbol{\Lambda}_i) = 0$

Discretized: $r_{\mathcal{E}_i}(\mathbf{J}_h,\mathbf{I}_h\ ) = a_{\mathcal{E},\mathcal{E}}(\mathbf{J}_h,\boldsymbol{\Lambda}_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_h\ ,\boldsymbol{\Lambda}_i) - b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}},\boldsymbol{\Lambda}_i) = 0$

Method of manufactured solutions modifies discretized equations:

$$\mathbf{r}_{\mathcal{E}}(\mathbf{J}_h,\mathbf{I}_h) = \mathbf{r}_{\mathcal{E}}(\mathbf{J}_{\text{MS}},\mathbf{I}_{\text{MS}})$$

$\mathbf{J}_{\text{MS}}$ and $\mathbf{I}_{\text{MS}}$ are manufactured solutions, $\mathbf{r}_{\mathcal{E}}(\mathbf{J}_{\text{MS}},\mathbf{I}_{\text{MS}})$ is computed exactly

New Discretized: $a_{\mathcal{E},\mathcal{E}}(\mathbf{J}_h,\boldsymbol{\Lambda}_i)+a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_h,\boldsymbol{\Lambda}_i) = \underbrace{a_{\mathcal{E},\mathcal{E}}(\mathbf{J}_{\text{MS}},\boldsymbol{\Lambda}_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_{\text{MS}},\boldsymbol{\Lambda}_i)}_{= b_{\mathcal{E}}\left(\mathbf{E}^{\mathcal{I}},\boldsymbol{\Lambda}_i\right):\ \text{implement via } \mathbf{E}^{\mathcal{I}}}$

$$\mathbf{E}^{\mathcal{I}}(\mathbf{x}) = \frac{j}{\epsilon\omega}\int_{S'}\left[k^2\mathbf{J}_{\text{MS}}(\mathbf{x}')G(\mathbf{x},\mathbf{x}') + \nabla'\cdot\mathbf{J}_{\text{MS}}(\mathbf{x}')\nabla G(\mathbf{x},\mathbf{x}')\right]dS' + Z_s\mathbf{J}_{\text{MS}}(\mathbf{x})$$
$$-\frac{1}{4}(\mathbf{n}(\mathbf{x})\times\mathbf{I}_{\text{MS}}(\mathbf{x}))\delta_{\text{slot}}(\mathbf{x}) + \frac{1}{4\pi}\int_0^L \mathbf{I}_{\text{MS}}(s')\times\int_0^{2\pi}\nabla' G(\mathbf{x},\mathbf{x}')d\phi' ds'$$

MMS incorporated through $\mathbf{E}^{\mathcal{I}}$ – no additional source term required

# Manufactured Solutions for the Slot Equation

Continuous: $r_{\mathcal{M}_i}(\mathbf{J}\ \ ,\mathbf{I}_m) = a_{\mathcal{M},\mathcal{E}}(\mathbf{J}\ \ ,\boldsymbol{\Lambda}_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_m,\boldsymbol{\Lambda}_i^m) = 0$

Discretized: $r_{\mathcal{M}_i}(\mathbf{J}_h,\mathbf{I}_h\ ) = a_{\mathcal{M},\mathcal{E}}(\mathbf{J}_h,\boldsymbol{\Lambda}_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_h\ ,\boldsymbol{\Lambda}_i^m) = 0$

Method of manufactured solutions modifies discretized equations:

$$\mathbf{r}_{\mathcal{M}}(\mathbf{J}_h,\mathbf{I}_h) = \mathbf{r}_{\mathcal{M}}(\mathbf{J}_{\text{MS}},\mathbf{I}_{\text{MS}})$$

New Discretized:

$$a_{\mathcal{M},\mathcal{E}}(\mathbf{J}_h,\boldsymbol{\Lambda}_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_h,\boldsymbol{\Lambda}_i^m) = \underbrace{a_{\mathcal{M},\mathcal{E}}(\mathbf{J}_{\text{MS}},\boldsymbol{\Lambda}_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_{\text{MS}},\boldsymbol{\Lambda}_i^m)}_{=0\text{: no source term needed}}$$

Given $\mathbf{J}_{\text{MS}}$, solve for $\mathbf{I}_m(s) = I_m(s)\mathbf{s}$ to avoid source term

$$J_s(s) = \sum_{q=1}^{\infty} J_{s_q} \sin\left(\frac{q\pi s}{L}\right), \qquad J_{s_q} = \frac{2}{L}\int_0^L J_s(s)\sin\left(\frac{q\pi s}{L}\right)ds,$$

$$I_m(s) = \sum_{q=1}^{\infty} I_{m_q} \sin\left(\frac{q\pi s}{L}\right), \qquad I_{m_q}^{\pm} = \frac{jw(k^2-\beta_x^2)}{\beta_{y_q}\omega\epsilon}\left(\left[J_{s_q}^+ + J_{s_q}^-\right]\tan(\beta_{y_q}d/2) \mp \left[J_{s_q}^+ - J_{s_q}^-\right]\cot(\beta_{y_q}d/2)\right)$$

# Solution-Discretization Error

- Error due to basis-function approximations of solutions:

$$\mathbf{J}_h(\mathbf{x}) = \sum_{j=1}^{n_b} J_j \mathbf{\Lambda}_j(\mathbf{x}), \qquad \mathbf{I}_h(s) = \sum_{j=1}^{n_b^m} I_j \mathbf{\Lambda}_j^m(s)$$

- Measured with discretization errors: $\mathbf{e_J} = \mathbf{J}^h - \mathbf{J}_n, \quad \mathbf{e_I} = \mathbf{I}^h - \mathbf{I}_s$

$$\|\mathbf{e_J}\| \leq C_\mathbf{J} h^{p_\mathbf{J}}, \qquad \|\mathbf{e_I}\| \leq C_\mathbf{I} h^{p_\mathbf{I}}$$

$J_{n_j}$: component of $\mathbf{J}_{\text{MS}}$ flowing from $T_j^+$ to $T_j^-$
$I_{s_j}$: component of $\mathbf{I}_{\text{MS}}$ flowing along $\mathbf{s}$ at $s_j$
$C$ : function of solution derivatives
$h$ : measure of mesh size
$p$ : order of accuracy

- Compute $p$ from $\|\mathbf{e}\|$ across multiple meshes (expect $p = 2$ for these bases)

- Avoid numerical-integration error if integrating exactly

# Solution-Discretization Error: Discontinuity

- $\delta_{\text{slot}}$ introduces discontinuity due to wire interaction with surface
- Discontinuity impacts $\mathbf{E}^{\mathcal{I}}$ for MMS
- Discontinuity will contaminate convergence studies: $\mathcal{O}(h^2) \rightarrow \mathcal{O}(h)$
- Discontinuity denoted by $\mathbf{B}_1$ in $\mathbf{Z} = \begin{bmatrix} \mathbf{A} & (\mathbf{B}_1 + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$
- Since $\mathbf{B}_1 = -\frac{1}{4}\mathbf{C}^T$, use $\mathbf{C}$ to cancel contribution from $\mathbf{B}_1$ and modify $\mathbf{E}^{\mathcal{I}}$:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{A} & (\cancel{\mathbf{B}_1} + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix},$$

$$\mathbf{E}^{\mathcal{I}} = \frac{j}{\epsilon\omega}\int_{S'}[k^2\mathbf{J}_{\text{MS}}G + \nabla'\cdot\mathbf{J}_{\text{MS}}\nabla G]dS' - \cancel{\frac{1}{4}(\mathbf{n}\times\mathbf{I}_{\text{MS}})\delta_{\text{slot}}} + \frac{1}{4\pi}\int_0^L \mathbf{I}_{\text{MS}} \times \int_0^{2\pi} \nabla' G d\phi' ds' + Z_s\mathbf{J}_{\text{MS}}$$

- Correctness of $\mathbf{B}_1$ is assessed by successful removal using $\mathbf{C}$
- Correctness of $\mathbf{C}$ is assessed through the mesh-convergence study

# Numerical-Integration Error

- Error due to quadrature integral evaluation $(\cdot)^q$ on both sides of equations
- Measure numerical-integration error:

$$e_a = \boldsymbol{\mathcal{J}}^H(\mathbf{Z}^q - \mathbf{Z})\boldsymbol{\mathcal{J}}, \qquad e_b = \boldsymbol{\mathcal{J}}^H(\mathbf{V}^q - \mathbf{V}),$$

where $\boldsymbol{\mathcal{J}} = \begin{Bmatrix} \mathbf{J}_n \\ \mathbf{I}_s \end{Bmatrix}$

- Solution-discretization error is canceled
- $|e_a| \le C_a h^{p_a}$ and $|e_b| \le C_b h^{p_b}$

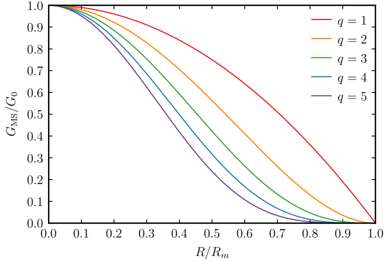
  $C$: function of integrand derivatives
  $p$: order of accuracy of quadrature rules
- With multiple meshes, compute $p$ from $|e|$

# Manufactured Green's Function

Integrals with $G$ cannot be computed analytically or, when $R \to 0$, accurately

Inaccurately computing integrals on either side contaminates convergence studies

Manufacture Green's function: $G_{\text{MS}}(R) = G_0\left(1 - \frac{R^2}{R_m^2}\right)^q$, $R_m = \max\limits_{\mathbf{x},\mathbf{x}' \in S} R$ and $q \in \mathbb{N}$




Reasoning:
1) Even powers of $R$ permit integrals to be computed analytically
2) $G_{\text{MS}}$ increases when $R$ decreases, as with actual $G$

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  - Domain and Parameters
  - Manufactured Surface Current
  - Magnetic Current
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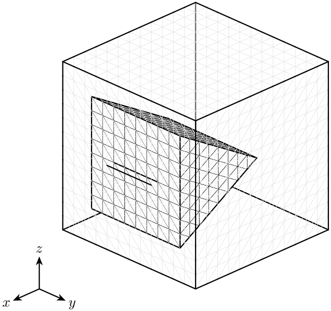
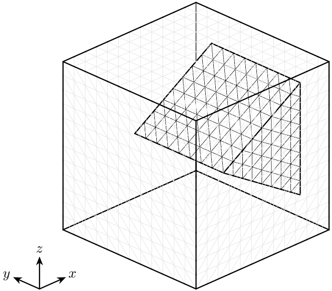
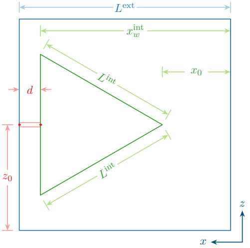
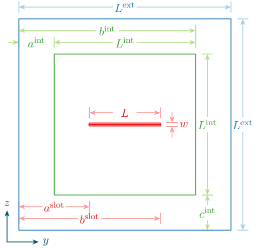
# Cubic Scatterer with a Triangularly Prismatic Cavity



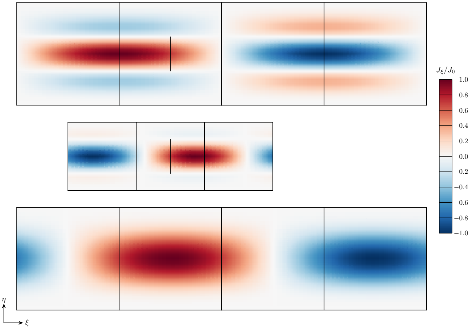



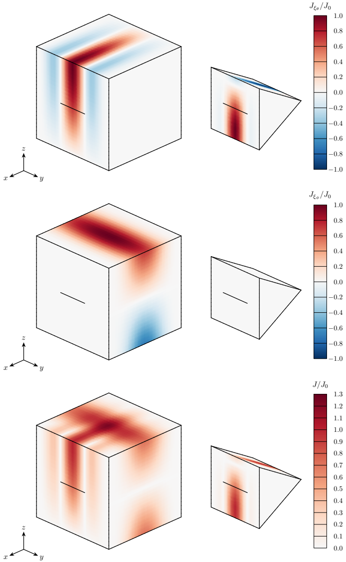




- $L^{\text{ext}} = 1$ m, $\quad L^{\text{int}} = 2L^{\text{ext}}/3, \quad L = L^{\text{ext}}/3, \quad w = L^{\text{ext}}/50$
- $a^{\text{int}} = L^{\text{ext}}/6, \quad c^{\text{int}} = L^{\text{ext}}/6, \quad z_0 = L^{\text{ext}}/2$
- $\mu = \mu_0, \quad \epsilon = \epsilon_0, \quad k = 2\pi \text{ m}^{-1}, \quad \sigma_{\text{scatterer}} \gg \sigma_{\text{slot}} > \sigma_0 = 0$
- 3 depths: $d_1 = L^{\text{ext}}/5, \quad d_2 = L^{\text{ext}}/10, \quad d_3 = L^{\text{ext}}/20$
- 2 Green's functions: $G_1$, $G_2$

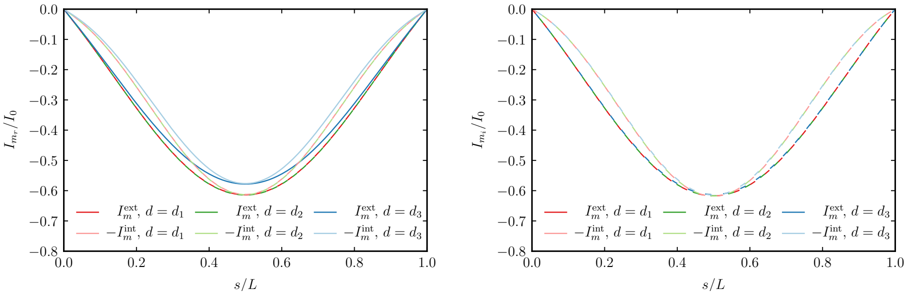
# Manufactured Surface Current




- Manufacture solutions for 2D strips of class $C^2$
- Wrap strips around lateral surfaces of prisms
- Solutions are product of $\xi$ and $\eta$ dependencies
  - $\xi$ dependency: sine function with single period
  - $\eta$ dependency: linear combination of odd-harmonic sine functions
- Current flows along $\xi$; at $s = \{0, L\}$, $J_\xi = 0$



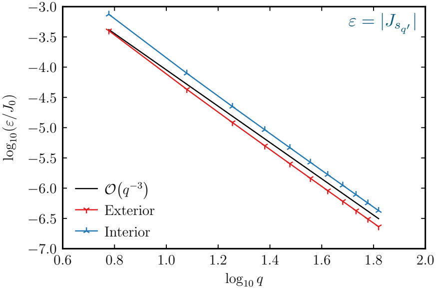
# Magnetic Current

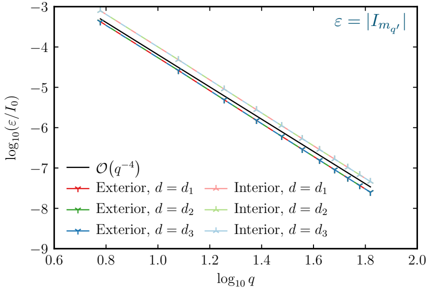



- For a given $\mathbf{J}_{\text{MS}}$, $\mathbf{I}_m(s) = I_m(s)\mathbf{s}$ is an infinite series that needs to be truncated

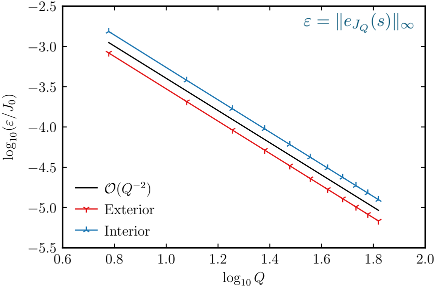
$$I_{m_Q}(s) = \sum_{q=1}^{Q} I_{m_{q'}} \sin\left(\frac{q'\pi s}{L}\right), \qquad q' = 2q - 1$$

# Sine Series Convergence

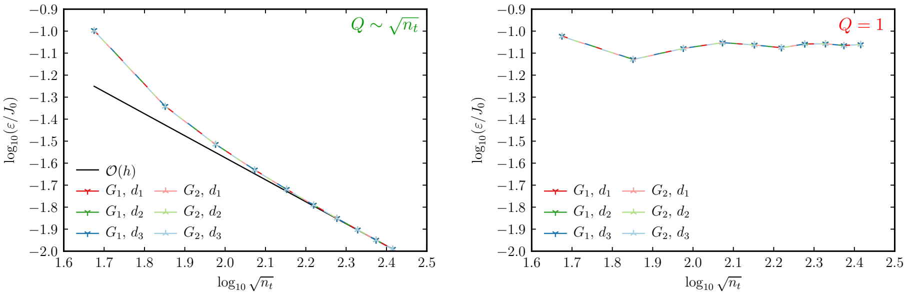



- $J_s$ sine coefficients are $\mathcal{O}(q^{-3})$
- $I_m$ sine coefficients are $\mathcal{O}(q^{-4})$
- $\|e_{J_Q}(s)\|_\infty = \max_{s\in[0,\,L]} |J_{s_Q}(s) - J_s(s)|$ is $\mathcal{O}(Q^{-2})$
- $\|e_{I_Q}(s)\|_\infty = \max_{s\in[0,\,L]} |I_{m_Q}(s) - I_m(s)|$ is $\mathcal{O}(Q^{-3})$
- Set $Q \sim \sqrt{n_t}$ to reduce truncation error faster
  - Basis-function error is $\mathcal{O}(h^2)$
  - $I_m$ truncation error is $\mathcal{O}(h^3)$



# Solution-Discretization Error: $\varepsilon = \|\mathbf{e_J}\|_\infty$ (with discontinuity)

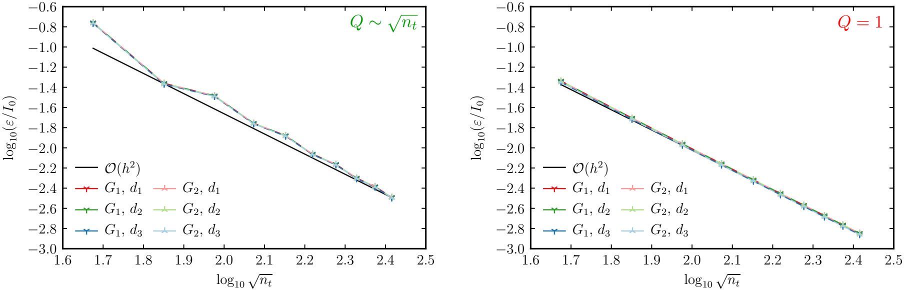



- Discontinuity present:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\mathcal{E} \\ \mathbf{0} \end{Bmatrix}$$

- Convergence rates for $Q = 1$ and $Q \sim \sqrt{n_t}$ are $\mathcal{O}(1)$ and $\mathcal{O}(h)$

# Solution-Discretization Error: $\varepsilon = \|\mathbf{e_I}\|_\infty$ (with discontinuity)

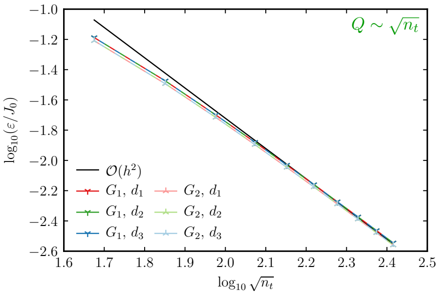


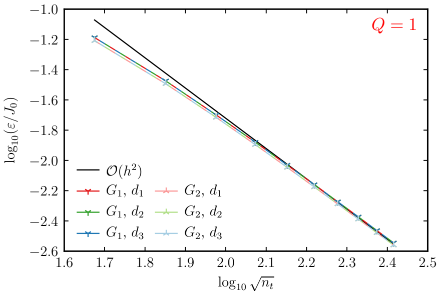

- Discontinuity present:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}} \\ \mathbf{0} \end{Bmatrix}$$

- Convergence rates for $Q = 1$ and $Q \sim \sqrt{n_t}$ are $\mathcal{O}(h^2)$

# Solution-Discretization Error: $\varepsilon = \|\mathbf{e_J}\|_\infty$ (without discontinuity)



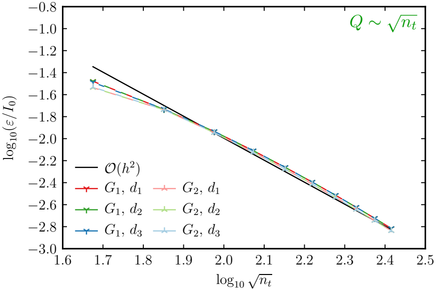



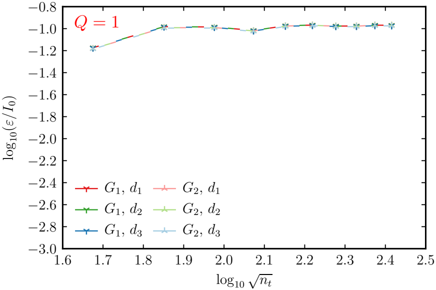

- Discontinuity removed from $\mathbf{Z}$ using $\mathbf{C}$, corresponding MMS source term omitted in $\mathbf{V}^\mathcal{E}$:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\mathcal{E} \\ \mathbf{0} \end{Bmatrix} \rightarrow \begin{bmatrix} \mathbf{A} & (\cancel{\mathbf{B}_1} + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\mathcal{E} \\ \mathbf{0} \end{Bmatrix}$$

- Convergence rates for $Q = 1$ and $Q \sim \sqrt{n_t}$ are $\mathcal{O}(h^2)$
- Correct implementation of $\mathbf{B}_1$ suggested by its removal using $\mathbf{C}$
- Correct implementation of $\mathbf{C}$ suggested by expected convergence rates

# Solution-Discretization Error: $\varepsilon = \|\mathbf{e_I}\|_\infty$ (without discontinuity)




- Discontinuity removed from $\mathbf{Z}$ using $\mathbf{C}$, corresponding MMS source term omitted in $\mathbf{V}^{\mathcal{E}}$:

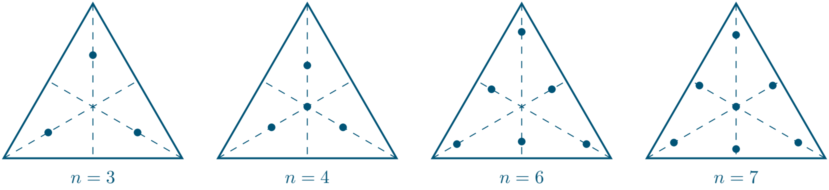
$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}} \\ \mathbf{0} \end{Bmatrix} \rightarrow \begin{bmatrix} \mathbf{A} & (\cancel{\mathbf{B}_1} + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}} \\ \mathbf{0} \end{Bmatrix}$$

- Convergence rates for $Q = 1$ and $Q \sim \sqrt{n_t}$ are $\mathcal{O}(h^2)$ and $\mathcal{O}(1)$
- Correct implementation of $\mathbf{B}_1$ suggested by its removal using $\mathbf{C}$
- Correct implementation of $\mathbf{C}$ suggested by expected convergence rates

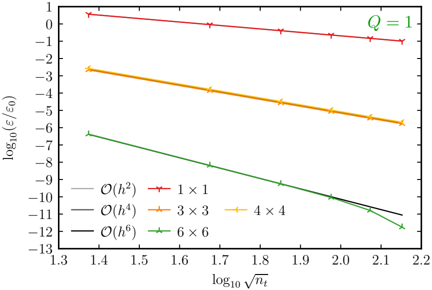
# Numerical Integration

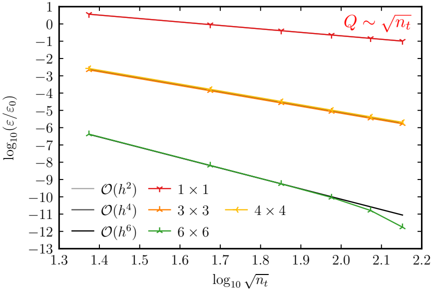
- Surface integrals evaluated using 2D triangle quadrature rules
- Wire integrals evaluated using 1D bar quadrature rules

| Maximum integrand degree | Number of 2D points | Number of 1D points | Convergence rate |
|---|---|---|---|
| 1 | 1 | 1 | $\mathcal{O}(h^2)$ |
| 2 | 3 | — | $\mathcal{O}(h^4)$ |
| 3 | 4 | 2 | $\mathcal{O}(h^4)$ |
| 4 | 6 | — | $\mathcal{O}(h^6)$ |
| 5 | 7 | 3 | $\mathcal{O}(h^6)$ |



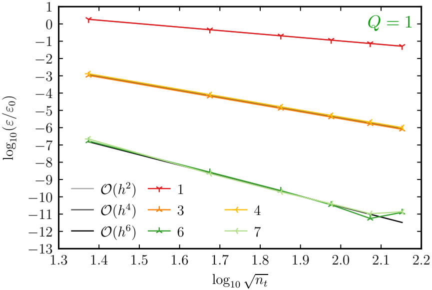
# Numerical-Integration Error: $\varepsilon = |e_a|$ ($G = G_2$, $d = d_1$)

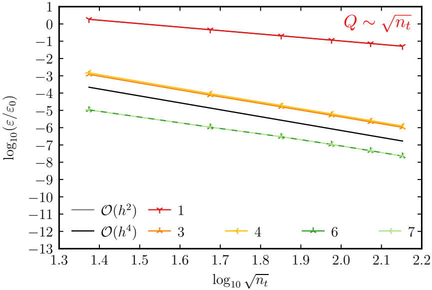



- 2D points: [number for test integral] × [number for source integral]
- 1D points: number of 1D points with same convergence rate as 2D points
- Convergence rates are as expected for $Q = 1$ and $Q \sim \sqrt{n_t}$

# Numerical-Integration Error: $\varepsilon = |e_b|$ $(G = G_2,\, d = d_1)$




- 2D points: number for test integral
- 1D points: number of 1D points with same convergence rate as 2D points
- Convergence rates are as expected for $Q = 1$
- Convergence rates are limited to $\mathcal{O}(h^4)$ for $Q \sim \sqrt{n_t}$ since $\left|\int_0^L (I_{m_Q}(s) - I_m(s))ds\right|$ is $\mathcal{O}(Q^{-4})$

# Outline

- Introduction
- Governing Equations
- Code-Verification Approaches
- Numerical Examples
- Summary
  - Closing Remarks

# Closing Remarks

3 error sources in electromagnetic integral equations:

- **Domain-discretization error** – avoided
  - Considered planar surfaces
- **Solution-discretization error** – isolated
  - Manufactured $\mathbf{J}$, chose $\mathbf{I}_m$ to avoid source term
  - Manufactured Green's function (to integrate exactly)
  - Removed discontinuity to measure convergence rates without contamination
  - Demonstrated implications of sine series truncation error on convergence
- **Numerical-integration error** – isolated
  - Removed solution-discretization error
  - Demonstrated implications of sine series truncation error on convergence

Achieved expected orders of accuracy

Questions? bafreno@sandia.gov brianfreno.github.io

# Additional Information


- B. Freno, N. Matula, W. Johnson
  Manufactured solutions for the method-of-moments implementation of the EFIE
  *Journal of Computational Physics* (2021) arXiv:2012.08681

- B. Freno, N. Matula, J. Owen, W. Johnson
  Code-verification techniques for the method-of-moments implementation of the EFIE
  *Journal of Computational Physics* (2022) arXiv:2106.13398

- B. Freno, N. Matula
  Code verification for practically singular equations
  *Journal of Computational Physics* (2022) arXiv:2204.01785

- B. Freno, N. Matula
  Code-verification techniques for the method-of-moments implementation of the MFIE
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  Code-verification techniques for the method-of-moments implementation of the CFIE
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- B. Freno, N. Matula, R. Pfeiffer, V. Dang
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  *Engineering Analysis with Boundary Elements* (2025) arXiv:2503.04004