

CODE-VERIFICATION TECHNIQUES FOR AN ARBITRARY-DEPTH ELECTROMAGNETIC SLOT MODEL

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Outline

- Introduction
- Governing Equations
- Code-Verification Approaches
- Numerical Examples
- Summary

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- Introduction
 - Electromagnetic Integral Equations
 - Verification and Validation
 - Error Sources
 - This Work
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Electromagnetic Integral Equations

- Are commonly used to model electromagnetic scattering and radiation
- Relate surface current to incident electric and/or magnetic field
- Discretize surface of electromagnetic scatterer with elements
- Evaluate 4D reaction integrals over 2D test and source elements
- Contain singular integrands when test and source elements are near

Electromagnetic Aperture and Slot Models

- EM penetration occurs through openings of otherwise closed surfaces
- Penetration may occur intentionally or unintentionally
- Slot connects exterior surface of scatterer to interior surface of cavity
- Model slot as wires carrying magnetic current located at apertures
 - Exterior surface interacts with exterior wire
 - Interior surface interacts with interior wire
 - Exterior and interior wires interact with each other
 - Exterior and interior surfaces do not interact directly

Verification and Validation

Credibility of computational physics codes requires verification and validation

- **Validation** assesses how well models represent physical phenomena
 - Compare computational results with experimental results
 - Assess suitability of models, model error, and bounds of validity
- **Verification** assesses accuracy of numerical solutions against expectations
 - *Solution verification* estimates numerical error for particular solution
 - *Code verification* verifies correctness of numerical-method implementation

Code Verification

- Code verification most rigorously assesses rate at which error decreases
- Error requires exact solution – usually unavailable
- Manufactured solutions are popular alternative
 - Manufacture an arbitrary solution
 - Insert manufactured solution into governing equations to get residual term
 - Add residual term to equations to coerce solution to manufactured solution
- For integral equations, few instances of code verification exist
- Analytical differentiation is straightforward – analytical integration is not
- Numerical integration is necessary, generally incurs an approximation error
- Therefore, manufactured source term may have its own numerical error

Error Sources in the Electromagnetic Integral Equations

3 sources of numerical error:

- **Domain discretization:** Representation of curved surfaces with planar elements
 - Second-order error for curved surfaces, no error for planar surfaces
 - Error reduced with curved elements
- **Solution discretization:** Representation of solution or operators
 - Common in solution to differential, integral, and integro-differential equations
 - Finite number of basis functions to approximate solution
 - Finite samples queried to approximate underlying equation operators
- **Numerical integration:** Quadrature
 - Analytical integration is not always possible
 - For well-behaved integrands,
 - Expect integration error at least same order as solution-discretization error
 - Less rigorously, error should decrease with more quadrature points
 - For (nearly) singular integrands, **monotonic convergence is not assured**

This Work

Isolate solution-discretization error

- Manufacture solution
- Eliminate numerical-integration error by manufacturing Green's function
- Mitigate contamination from discontinuity due to wire-surface interaction

Isolate numerical-integration error

- Manufacture solution
- Cancel solution-discretization error using basis functions

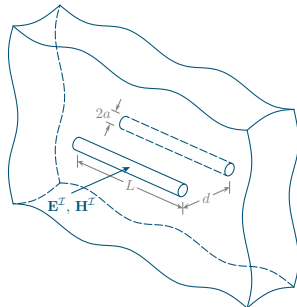
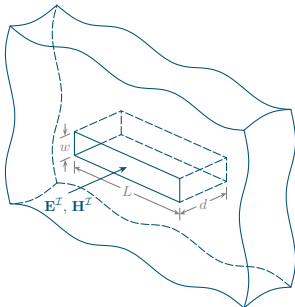
Avoid domain-discretization error

- Consider only planar surfaces
- Previously provided approaches to account for domain-discretization error

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- **Governing Equations**
 - Overview
 - The Electric-Field Integral Equation
 - The Slot Equation
 - Discretization
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Arbitrary-Depth Slot Model Overview



- Electromagnetic scatterer of arbitrary depth encloses a cavity
- Exterior is connected to interior by rectangularly prismatic slot with $L \gg w$ (left)
- Slot is replaced with two thin wires at apertures that carry magnetic current (right)
- Exterior and interior surfaces interact with wires, not each other
- Wires interact with each other through waveguide model
- EFIE solved on each surface, slot equation solved for wires

The Electric-Field Integral Equation

In time-harmonic form, \mathbf{E}^S computed from \mathbf{J} and \mathbf{M}

$$\text{Scattered electric field} \quad \mathbf{E}^S(\mathbf{x}) = -\left(j\omega\mathbf{A}(\mathbf{x}) + \nabla\Phi(\mathbf{x}) + \frac{1}{\epsilon}\nabla \times \mathbf{F}(\mathbf{x})\right)$$

$$\text{Magnetic vector potential} \quad \mathbf{A}(\mathbf{x}) = \mu \int_{S'} \mathbf{J}(\mathbf{x}')G(\mathbf{x}, \mathbf{x}')dS'$$

$$\text{Electric scalar potential} \quad \Phi(\mathbf{x}) = \frac{j}{\epsilon\omega} \int_{S'} \nabla' \cdot \mathbf{J}(\mathbf{x}')G(\mathbf{x}, \mathbf{x}')dS'$$

$$\text{Electric vector potential} \quad \mathbf{F}(\mathbf{x}) = \epsilon \int_{S'} \mathbf{M}(\mathbf{x}')G(\mathbf{x}, \mathbf{x}')dS'$$

$$\text{Green's function} \quad G(\mathbf{x}, \mathbf{x}') = \frac{e^{-jkR}}{4\pi R}, \quad R = |\mathbf{x} - \mathbf{x}'|$$

Singularity when $R \rightarrow 0$

\mathbf{J} and \mathbf{M} are electric and magnetic surface current densities

$S' = S$ is surface of scatterer

μ and ϵ are permeability and permittivity of surrounding medium

$k = \omega\sqrt{\mu\epsilon}$ is wavenumber

The Electric-Field Integral Equation (continued)

Compute \mathbf{J} and \mathbf{M} from incident electric field $\mathbf{E}^{\mathcal{I}}$ ($\mathbf{n} \times (\mathbf{E}^{\mathcal{S}} + \mathbf{E}^{\mathcal{I}}) = Z_s \mathbf{n} \times \mathbf{J}$)

Discretize surface with triangles, approximate \mathbf{J} with RWG basis functions:

$$\mathbf{J}_h(\mathbf{x}) = \sum_{j=1}^{n_b} J_j \mathbf{\Lambda}_j(\mathbf{x})$$

Project EFIE onto vector-valued RWG basis functions

Express \mathbf{M} in terms of filament magnetic current \mathbf{I}_m

Discretize wire with bars, approximate \mathbf{I}_m with 1D basis functions:

$$\mathbf{I}_h(s) = \sum_{j=1}^{n_b^m} I_j \mathbf{\Lambda}_j^m(s)$$

The Slot Equation

The magnetic current along the slot is modeled using a waveguide model:

$$\mathbf{s} \cdot (\mathbf{J}^{\pm} \times \mathbf{n}^{\pm}) + \frac{j\omega\epsilon}{2wL(k^2 - \beta_x^2)} \sum_{p=1}^{\infty} \beta_{y_p} \int_0^L \sin\left(\frac{p\pi s}{L}\right) \sin\left(\frac{p\pi s'}{L}\right) \times$$

$$(\pm [I_m^-(s') - I_m^+(s')] \tan(\beta_{y_p} d/2) + [I_m^+(s') + I_m^-(s')] \cot(\beta_{y_p} d/2)) ds' = 0$$

$$I_m(0) = I_m(L) = 0$$

\mathbf{s} is the direction of the wire ($\mathbf{I}_m = I_m(s)\mathbf{s}$)

Superscripts – and + denote exterior and interior

Effective wire radius a obtained from conformal mapping using w and d

β_{α} is the propagation constant in the α direction

Project slot equation onto vector-valued 1D basis functions

Discretized Problem

Find $\mathbf{J}_h \in \mathbb{V}_h$ and $\mathbf{I}_h \in \mathbb{V}_h^m$, such that

$$a_{\mathcal{E},\mathcal{E}}(\mathbf{J}_h, \mathbf{\Lambda}_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_h, \mathbf{\Lambda}_i) = b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}}, \mathbf{\Lambda}_i) \quad \text{for } i = 1, \dots, n_b \quad (\text{EFIE})$$

$$a_{\mathcal{M},\mathcal{E}}(\mathbf{J}_h, \mathbf{\Lambda}_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_h, \mathbf{\Lambda}_i^m) = 0 \quad \text{for } i = 1, \dots, n_b^m \quad (\text{Slot})$$

Evaluate EFIE on exterior and interior surfaces: $n_b^- + n_b^+$ unknowns for \mathbf{J}_h

Evaluate slot equation on exterior and interior wires: $n_b^{m-} + n_b^{m+}$ unknowns for \mathbf{I}_h

Matrix–Vector Form

In matrix–vector form, solve for \mathcal{J}^h :

$$\mathbf{Z}\mathcal{J}^h = \mathbf{V}$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{A}^- & \mathbf{0} & \mathbf{B}^- & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^+ & \mathbf{0} & \mathbf{B}^+ \\ \mathbf{C}^- & \mathbf{0} & \mathbf{D}_{\sim}^- & \mathbf{D}_{\not\sim}^- \\ \mathbf{0} & \mathbf{C}^+ & \mathbf{D}_{\not\sim}^+ & \mathbf{D}_{\sim}^+ \end{bmatrix}, \quad \mathcal{J}^h = \begin{Bmatrix} \mathbf{J}^{h-} \\ \mathbf{J}^{h+} \\ \mathbf{I}^{h-} \\ \mathbf{I}^{h+} \end{Bmatrix}, \quad \mathbf{V} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}-} \\ \mathbf{V}^{\mathcal{E}+} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix},$$

Impedance matrix Current vector Excitation vector

where

$$A_{i,j} = a_{\mathcal{E},\mathcal{E}}(\boldsymbol{\Lambda}_j, \boldsymbol{\Lambda}_i), \quad B_{i,j} = a_{\mathcal{E},\mathcal{M}}(\boldsymbol{\Lambda}_j^m, \boldsymbol{\Lambda}_i), \quad C_{i,j} = a_{\mathcal{M},\mathcal{E}}(\boldsymbol{\Lambda}_j, \boldsymbol{\Lambda}_i^m), \quad D_{i,j} = a_{\mathcal{M},\mathcal{M}}(\boldsymbol{\Lambda}_j^m, \boldsymbol{\Lambda}_i^m),$$

$$J_j^h = J_j, \quad I_j^h = I_j, \quad V_j^{\mathcal{E}} = b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}}, \boldsymbol{\Lambda}_i)$$

More compactly:
$$\mathbf{Z} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \quad \mathcal{J}^h = \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix}, \quad \mathbf{V} = \begin{Bmatrix} \mathbf{V}^{\mathcal{E}} \\ \mathbf{0} \end{Bmatrix}$$

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 - Numerical-Integration Error
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Manufactured Solutions for the EFIE

Continuous: $r_{\mathcal{E}_i}(\mathbf{J}, \mathbf{I}_m) = a_{\mathcal{E},\mathcal{E}}(\mathbf{J}, \Lambda_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_m, \Lambda_i) - b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}}, \Lambda_i) = 0$

Discretized: $r_{\mathcal{E}_i}(\mathbf{J}_h, \mathbf{I}_h) = a_{\mathcal{E},\mathcal{E}}(\mathbf{J}_h, \Lambda_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_h, \Lambda_i) - b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}}, \Lambda_i) = 0$

Method of manufactured solutions modifies discretized equations:

$$\mathbf{r}_{\mathcal{E}}(\mathbf{J}_h, \mathbf{I}_h) = \mathbf{r}_{\mathcal{E}}(\mathbf{J}_{\text{MS}}, \mathbf{I}_{\text{MS}})$$

\mathbf{J}_{MS} and \mathbf{I}_{MS} are manufactured solutions, $\mathbf{r}_{\mathcal{E}}(\mathbf{J}_{\text{MS}}, \mathbf{I}_{\text{MS}})$ is computed exactly

New Discretized: $a_{\mathcal{E},\mathcal{E}}(\mathbf{J}_h, \Lambda_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_h, \Lambda_i) = \underbrace{a_{\mathcal{E},\mathcal{E}}(\mathbf{J}_{\text{MS}}, \Lambda_i) + a_{\mathcal{E},\mathcal{M}}(\mathbf{I}_{\text{MS}}, \Lambda_i)}_{= b_{\mathcal{E}}(\mathbf{E}^{\mathcal{I}}, \Lambda_i): \text{implement via } \mathbf{E}^{\mathcal{I}}}$

$$\begin{aligned} \mathbf{E}^{\mathcal{I}}(\mathbf{x}) = & \frac{j}{\epsilon\omega} \int_{S'} [k^2 \mathbf{J}_{\text{MS}}(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') + \nabla' \cdot \mathbf{J}_{\text{MS}}(\mathbf{x}') \nabla G(\mathbf{x}, \mathbf{x}')] dS' + Z_s \mathbf{J}_{\text{MS}}(\mathbf{x}) \\ & - \frac{1}{4} (\mathbf{n}(\mathbf{x}) \times \mathbf{I}_{\text{MS}}(\mathbf{x})) \delta_{\text{slot}}(\mathbf{x}) + \frac{1}{4\pi} \int_0^L \mathbf{I}_{\text{MS}}(s') \times \int_0^{2\pi} \nabla' G(\mathbf{x}, \mathbf{x}') d\phi' ds' \end{aligned}$$

MMS incorporated through $\mathbf{E}^{\mathcal{I}}$ – no additional source term required

Manufactured Solutions for the Slot Equation

Continuous: $r_{\mathcal{M}_i}(\mathbf{J}, \mathbf{I}_m) = a_{\mathcal{M},\mathcal{E}}(\mathbf{J}, \Lambda_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_m, \Lambda_i^m) = 0$

Discretized: $r_{\mathcal{M}_i}(\mathbf{J}_h, \mathbf{I}_h) = a_{\mathcal{M},\mathcal{E}}(\mathbf{J}_h, \Lambda_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_h, \Lambda_i^m) = 0$

Method of manufactured solutions modifies discretized equations:

$$\mathbf{r}_{\mathcal{M}}(\mathbf{J}_h, \mathbf{I}_h) = \mathbf{r}_{\mathcal{M}}(\mathbf{J}_{\text{MS}}, \mathbf{I}_{\text{MS}})$$

New **Discretized:**

$$a_{\mathcal{M},\mathcal{E}}(\mathbf{J}_h, \Lambda_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_h, \Lambda_i^m) = \underbrace{a_{\mathcal{M},\mathcal{E}}(\mathbf{J}_{\text{MS}}, \Lambda_i^m) + a_{\mathcal{M},\mathcal{M}}(\mathbf{I}_{\text{MS}}, \Lambda_i^m)}_{= 0: \text{ no source term needed}}$$

Given \mathbf{J}_{MS} , solve for $\mathbf{I}_m(s) = I_m(s)\mathbf{s}$ to avoid source term

$$J_s(s) = \sum_{q=1}^{\infty} J_{s_q} \sin\left(\frac{q\pi s}{L}\right), \quad J_{s_q} = \frac{2}{L} \int_0^L J_s(s) \sin\left(\frac{q\pi s}{L}\right) ds,$$

$$I_m(s) = \sum_{q=1}^{\infty} I_{m_q} \sin\left(\frac{q\pi s}{L}\right), \quad I_{m_q}^{\pm} = \frac{jw(k^2 - \beta_x^2)}{\beta_{y_q} \omega \epsilon} ([J_{s_q}^+ + J_{s_q}^-] \tan(\beta_{y_q} d/2) \mp [J_{s_q}^+ - J_{s_q}^-] \cot(\beta_{y_q} d/2))$$

Solution-Discretization Error

- Error due to basis-function approximations of solutions:

$$\mathbf{J}_h(\mathbf{x}) = \sum_{j=1}^{n_b} J_j \Lambda_j(\mathbf{x}), \quad \mathbf{I}_h(s) = \sum_{j=1}^{n_b^m} I_j \Lambda_j^m(s)$$

- Measured with discretization errors: $\mathbf{e}_J = \mathbf{J}^h - \mathbf{J}_n$, $\mathbf{e}_I = \mathbf{I}^h - \mathbf{I}_s$

$$\|\mathbf{e}_J\| \leq C_J h^{p_J}, \quad \|\mathbf{e}_I\| \leq C_I h^{p_I}$$

J_{n_j} : component of \mathbf{J}_{MS} flowing from T_j^+ to T_j^-

I_{s_j} : component of \mathbf{I}_{MS} flowing along \mathbf{s} at s_j

C : function of solution derivatives

h : measure of mesh size

p : order of accuracy

- Compute p from $\|\mathbf{e}\|$ across multiple meshes (expect $p = 2$ for these bases)
- Avoid numerical-integration error if integrating exactly

Solution-Discretization Error: Discontinuity

- δ_{slot} introduces discontinuity due to wire interaction with surface
- Discontinuity impacts $\mathbf{E}^{\mathcal{I}}$ for MMS
- Discontinuity will contaminate convergence studies: $\mathcal{O}(h^2) \rightarrow \mathcal{O}(h)$
- Discontinuity denoted by \mathbf{B}_1 in $\mathbf{Z} = \begin{bmatrix} \mathbf{A} & (\mathbf{B}_1 + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$
- Since $\mathbf{B}_1 = -\frac{1}{4}\mathbf{C}^T$, use \mathbf{C} to cancel contribution from \mathbf{B}_1 and modify $\mathbf{E}^{\mathcal{I}}$:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{A} & (\cancel{\mathbf{B}_1} + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix},$$

$$\mathbf{E}^{\mathcal{I}} = \frac{j}{\epsilon\omega} \int_{S'} [k^2 \mathbf{J}_{\text{MS}} G + \nabla' \cdot \mathbf{J}_{\text{MS}} \nabla G] dS' - \frac{1}{4} (\mathbf{n} \times \mathbf{I}_{\text{MS}}) \delta_{\text{slot}} + \frac{1}{4\pi} \int_0^L \mathbf{I}_{\text{MS}} \times \int_0^{2\pi} \nabla' G d\phi' ds' + Z_s \mathbf{J}_{\text{MS}}$$

- Correctness of \mathbf{B}_1 is assessed by successful removal using \mathbf{C}
- Correctness of \mathbf{C} is assessed through the mesh-convergence study

Numerical-Integration Error

- Error due to quadrature integral evaluation $(\cdot)^q$ on both sides of equations
- Measure numerical-integration error:

$$e_a = \mathcal{J}^H(\mathbf{Z}^q - \mathbf{Z})\mathcal{J}, \quad e_b = \mathcal{J}^H(\mathbf{V}^q - \mathbf{V}),$$

where $\mathcal{J} = \begin{Bmatrix} \mathbf{J}_n \\ \mathbf{I}_s \end{Bmatrix}$

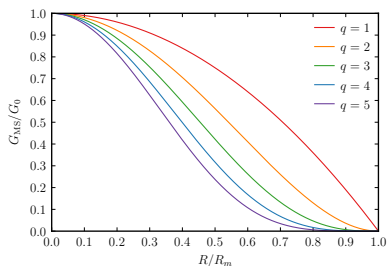
- Solution-discretization error is canceled
- $|e_a| \leq C_a h^{p_a}$ and $|e_b| \leq C_b h^{p_b}$
 C : function of integrand derivatives
 p : order of accuracy of quadrature rules
- With multiple meshes, compute p from $|e|$

Manufactured Green's Function

Integrals with G cannot be computed analytically or, when $R \rightarrow 0$, accurately

Inaccurately computing integrals on either side contaminates convergence studies

Manufacture Green's function: $G_{MS}(R) = G_0 \left(1 - \frac{R^2}{R_m^2}\right)^q$, $R_m = \max_{\mathbf{x}, \mathbf{x}' \in S} R$ and $q \in \mathbb{N}$



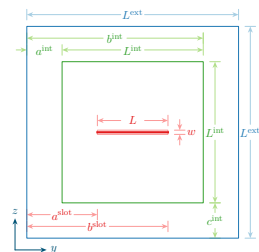
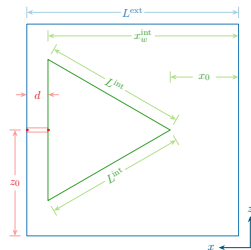
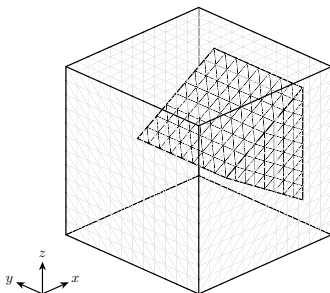
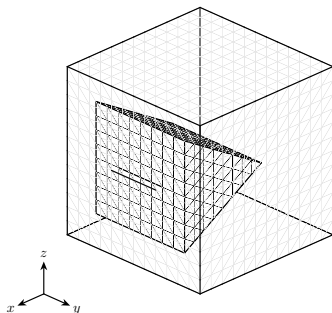
Reasoning:

- 1) Even powers of R permit integrals to be computed analytically
- 2) G_{MS} increases when R decreases, as with actual G

Outline

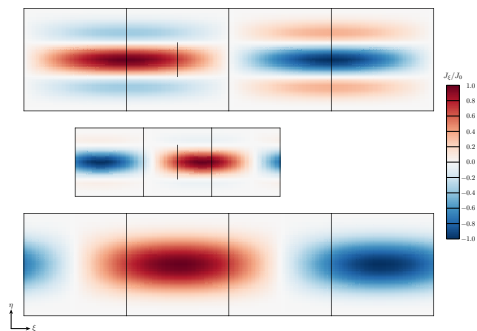
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Cubic Scatterer with a Triangularly Prismatic Cavity

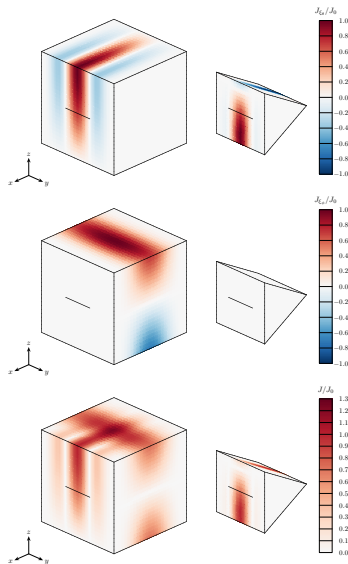


- $L^{\text{ext}} = 1 \text{ m}$, $L^{\text{int}} = 2L^{\text{ext}}/3$, $L = L^{\text{ext}}/3$, $w = L^{\text{ext}}/50$
- $a^{\text{int}} = L^{\text{ext}}/6$, $c^{\text{int}} = L^{\text{ext}}/6$, $z_0 = L^{\text{ext}}/2$
- $\mu = \mu_0$, $\epsilon = \epsilon_0$, $k = 2\pi \text{ m}^{-1}$, $\sigma_{\text{scatterer}} \gg \sigma_{\text{slot}} > \sigma_0 = 0$
- 3 depths: $d_1 = L^{\text{ext}}/5$, $d_2 = L^{\text{ext}}/10$, $d_3 = L^{\text{ext}}/20$
- 2 Green's functions: G_1, G_2

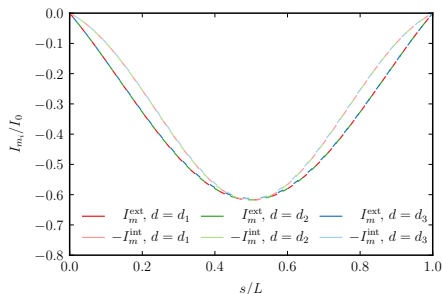
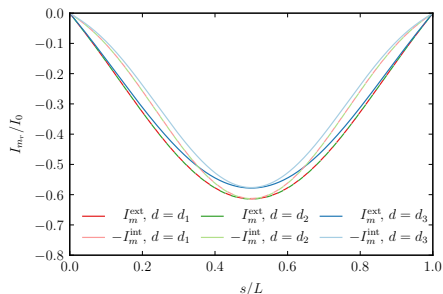
Manufactured Surface Current



- Manufacture solutions for 2D strips of class C^2
- Wrap strips around lateral surfaces of prisms
- Solutions are product of ξ and η dependencies
 - ξ dependency: sine function with single period
 - η dependency: linear combination of odd-harmonic sine functions
- Current flows along ξ ; at $s = \{0, L\}$, $J_\xi = 0$



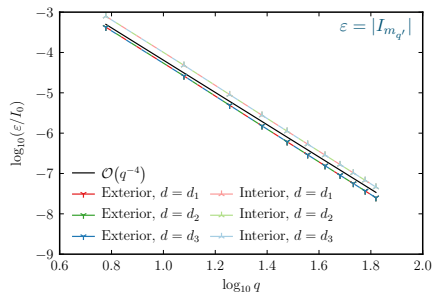
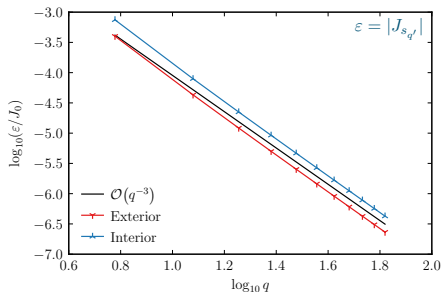
Magnetic Current



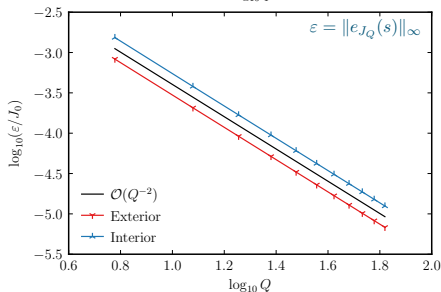
- For a given \mathbf{J}_{MS} , $\mathbf{I}_m(s) = I_m(s)\mathbf{s}$ is an infinite series that needs to be truncated

$$I_{mQ}(s) = \sum_{q=1}^Q I_{m_{q'}} \sin\left(\frac{q'\pi s}{L}\right), \quad q' = 2q - 1$$

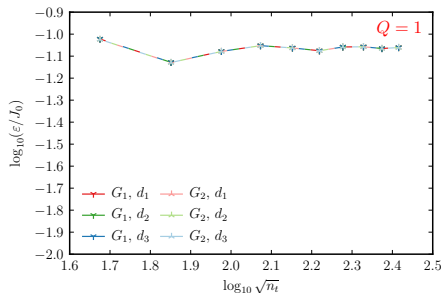
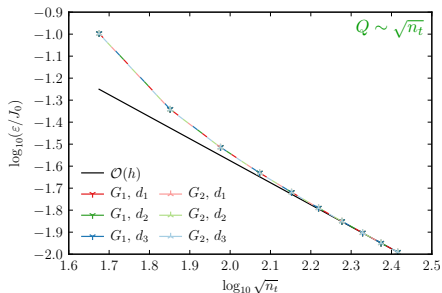
Sine Series Convergence



- J_s sine coefficients are $\mathcal{O}(q^{-3})$
- I_m sine coefficients are $\mathcal{O}(q^{-4})$
- $\|e_{J_Q}(s)\|_\infty = \max_{s \in [0, L]} |J_{s_Q}(s) - J_s(s)|$ is $\mathcal{O}(Q^{-2})$
- $\|e_{I_Q}(s)\|_\infty = \max_{s \in [0, L]} |I_{m_Q}(s) - I_m(s)|$ is $\mathcal{O}(Q^{-3})$
- Set $Q \sim \sqrt{n_t}$ to reduce truncation error faster
 - Basis-function error is $\mathcal{O}(h^2)$
 - I_m truncation error is $\mathcal{O}(h^3)$



Solution-Discretization Error: $\varepsilon = \|\mathbf{e}_J\|_\infty$ (with discontinuity)

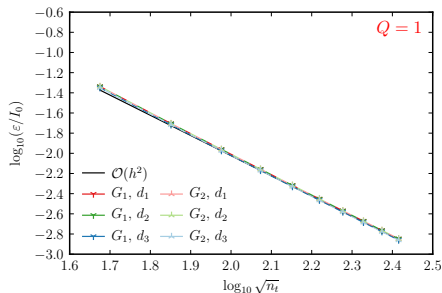
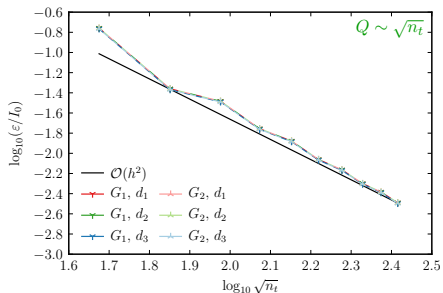


- Discontinuity present:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\varepsilon \\ \mathbf{0} \end{Bmatrix}$$

- Convergence rates for $Q = 1$ and $Q \sim \sqrt{n_t}$ are $\mathcal{O}(1)$ and $\mathcal{O}(h)$

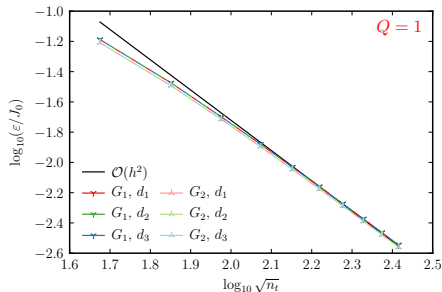
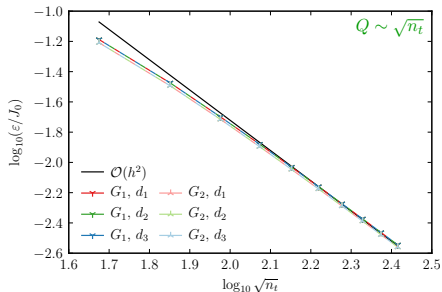
Solution-Discretization Error: $\varepsilon = \|\mathbf{e}_I\|_\infty$ (with discontinuity)



- Discontinuity present:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\varepsilon \\ \mathbf{0} \end{Bmatrix}$$

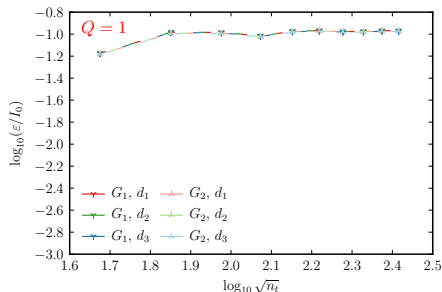
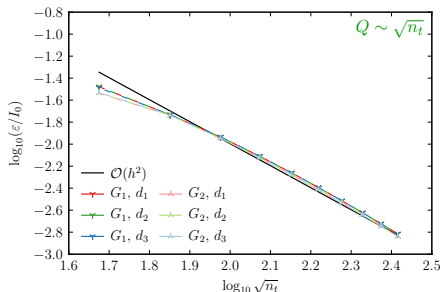
- Convergence rates for $Q = 1$ and $Q \sim \sqrt{n_t}$ are $\mathcal{O}(h^2)$

Solution-Discretization Error: $\varepsilon = \|\mathbf{e}_J\|_\infty$ (without discontinuity)

- Discontinuity removed from \mathbf{Z} using \mathbf{C} , corresponding MMS source term omitted in \mathbf{V}^ε :

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\varepsilon \\ \mathbf{0} \end{Bmatrix} \rightarrow \begin{bmatrix} \mathbf{A} & (\mathbf{B}_1 + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\varepsilon \\ \mathbf{0} \end{Bmatrix}$$

- Convergence rates for $Q = 1$ and $Q \sim \sqrt{n_t}$ are $\mathcal{O}(h^2)$
- Correct implementation of \mathbf{B}_1 suggested by its removal using \mathbf{C}
- Correct implementation of \mathbf{C} suggested by expected convergence rates

Solution-Discretization Error: $\varepsilon = \|\mathbf{e}_I\|_\infty$ (without discontinuity)

- Discontinuity removed from \mathbf{Z} using \mathbf{C} , corresponding MMS source term omitted in \mathbf{V}^ε :

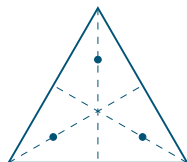
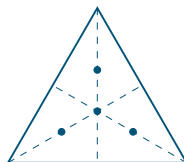
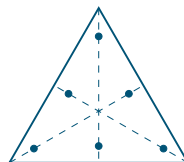
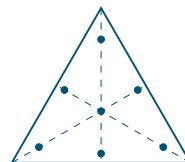
$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\varepsilon \\ \mathbf{0} \end{Bmatrix} \rightarrow \begin{bmatrix} \mathbf{A} & (\mathbf{B}_1 + \mathbf{B}_2) \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{J}^h \\ \mathbf{I}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\varepsilon \\ \mathbf{0} \end{Bmatrix}$$

- Convergence rates for $Q = 1$ and $Q \sim \sqrt{n_t}$ are $\mathcal{O}(h^2)$ and $\mathcal{O}(1)$
- Correct implementation of \mathbf{B}_1 suggested by its removal using \mathbf{C}
- Correct implementation of \mathbf{C} suggested by expected convergence rates

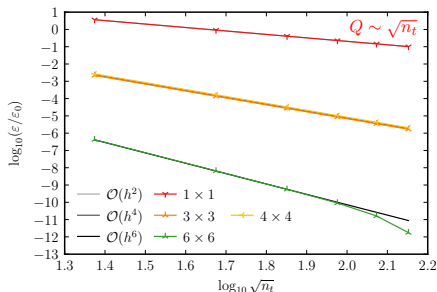
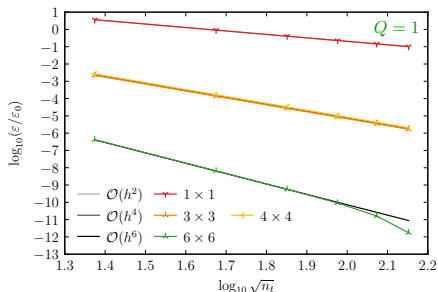
Numerical Integration

- Surface integrals evaluated using 2D triangle quadrature rules
- Wire integrals evaluated using 1D bar quadrature rules

Maximum integrand degree	Number of 2D points	Number of 1D points	Convergence rate
1	1	1	$\mathcal{O}(h^2)$
2	3	—	$\mathcal{O}(h^4)$
3	4	2	$\mathcal{O}(h^4)$
4	6	—	$\mathcal{O}(h^6)$
5	7	3	$\mathcal{O}(h^6)$

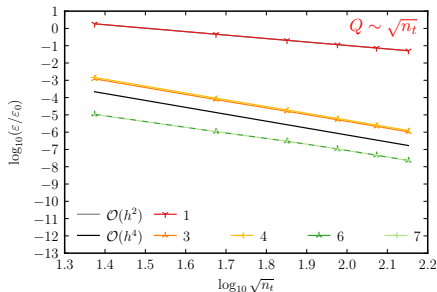
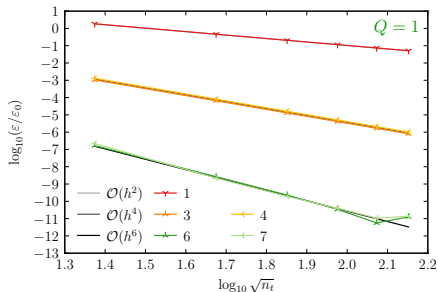
 $n = 3$  $n = 4$  $n = 6$  $n = 7$

Numerical-Integration Error: $\varepsilon = |e_a|$ ($G = G_2, d = d_1$)



- 2D points: [number for test integral] \times [number for source integral]
- 1D points: number of 1D points with same convergence rate as 2D points
- Convergence rates are as expected for $Q = 1$ and $Q \sim \sqrt{n_t}$

Numerical-Integration Error: $\varepsilon = |e_b|$ ($G = G_2, d = d_1$)



- 2D points: number for test integral
- 1D points: number of 1D points with same convergence rate as 2D points
- Convergence rates are as expected for $Q = 1$
- Convergence rates are limited to $\mathcal{O}(h^4)$ for $Q \sim \sqrt{n_t}$ since $\left| \int_0^L (I_{m_Q}(s) - I_m(s)) ds \right|$ is $\mathcal{O}(Q^{-4})$

Outline

- Introduction
- Governing Equations
- Code-Verification Approaches
- Numerical Examples
- **Summary**
 - Closing Remarks

Closing Remarks

3 error sources in electromagnetic integral equations:

- **Domain-discretization error** – avoided
 - Considered planar surfaces
- **Solution-discretization error** – isolated
 - Manufactured \mathbf{J} , chose \mathbf{I}_m to avoid source term
 - Manufactured Green's function (to integrate exactly)
 - Removed discontinuity to measure convergence rates without contamination
 - Demonstrated implications of sine series truncation error on convergence
- **Numerical-integration error** – isolated
 - Removed solution-discretization error
 - Demonstrated implications of sine series truncation error on convergence

Achieved expected orders of accuracy

Additional Information

- B. Freno, N. Matula, W. Johnson
Manufactured solutions for the method-of-moments implementation of the EFIE
Journal of Computational Physics (2021) [arXiv:2012.08681](#)
- B. Freno, N. Matula, J. Owen, W. Johnson
Code-verification techniques for the method-of-moments implementation of the EFIE
Journal of Computational Physics (2022) [arXiv:2106.13398](#)
- B. Freno, N. Matula
Code verification for practically singular equations
Journal of Computational Physics (2022) [arXiv:2204.01785](#)
- B. Freno, N. Matula
Code-verification techniques for the method-of-moments implementation of the MFIE
Journal of Computational Physics (2023) [arXiv:2209.09378](#)
- B. Freno, N. Matula
Code-verification techniques for the method-of-moments implementation of the CFIE
Journal of Computational Physics (2023) [arXiv:2302.06728](#)
- B. Freno, N. Matula, R. Pfeiffer, E. Dohme, J. Kotulski
Manufactured solutions for an electromagnetic slot model
Journal of Computational Physics (2024) [arXiv:2406.14573](#)
- B. Freno, N. Matula, R. Pfeiffer, V. Dang
Code-verification techniques for an arbitrary-depth electromagnetic slot model
[arXiv:2503.04004](#)

